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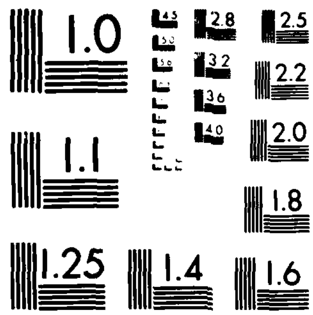
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APPROXIMATION SCHEMES FOR VISCOSITY SOLUTIONS OF *HAMILTON-JACOBI EQUATIONS*

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ABSTRACT

Equations of Hamilton-Jacobi type arise in many areas of application, including the calculus of variations, control theory and differential games. Recently M. G. Crandall and P. L. Lions established the correct notion of generalized solutions for these equations. This article discusses the convergence of general approximation schemes to this solution and gives, under certain hypotheses, explicit error estimates. These results are then applied to obtain various representations. These include "max-min" representations of solutions relevant to the theory of differential games (which imply the existence of the "value" of the game), representations as limits of solutions of general explicit and implicit finite difference schemes, and as limits of several types of Trotter products.

AMS (MOS) Subject Classifications: 35F20, 35F25, 35L60, 65M15, 65M10, 49A40, 49A45, 49B40, 90D25

Key Words: Hamilton-Jacobi equations, approximation schemes, error estimates, differential games, min-max representations, finite difference schemes, Trotter products

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SIGNIFICANCE AND EXPLANATION

Equations of Hamilton-Jacobi type arise in many area of application, including the calculus of variations, control theory and differential games. However, nonlinear first order partial differential equations almost never have global solutions, and one must deal with generalized solutions. Recently M. G. Crandall and P. L. Lions established the class of <u>viscosity</u> solutions of equations of Hamilton-Jacobi type and proved uniqueness within this class. Moreover, several results concerning the existence of this solution were given by M. G. Crandall and P. L. Lions, P. L.Lions, G. Barles and the author. This paper discusses general approximation schemes with applications for the viscosity solution and gives, under certain hypotheses, explicit error estimates. These results are then applied to obtain various representations. These include "max-min" representations of solutions relevant to the theory of differential games (which imply the existence of the "value" of the game). In particular, under certain assumptions, the solutions can be represented as the uniform limit of repeated min-max operations on the solutions of linear problems. Other representations include limits of solutions of general explicit and implicit finite difference schemes (with error estimates), and limits of Trotter products.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

APPROXIMATION SCHEMES FOR VISCOSITY
SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

Panagiotis E. Souganidis

INTRODUCTION

Recently M. G. Crandall and P. L. Lions ([5], also see M. G. Crandall, L. C. Evans and P. L. Lions [4]) introduced the notion of the viscosity solution of nonlinear first order partial differential equations. They used this notion to prove uniqueness and stability results for the Hamilton-Jacobi type equations, in particular the initial value problem

$$(0.1)\qquad \begin{cases} \frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N \end{cases}$$

and the stationary problem

$$(0.2)\qquad u + \lambda H(x,u,Du) = v \quad \text{in} \quad \mathbb{R}^N$$

where $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous and $Du = (\partial u/\partial y_1,\ldots,\partial u/\partial y_N)$ denotes the gradient of u. The existence of this solution for the problems (0.1) and (0.2) was established by M. G. Crandall and P. L. Lions [5], P. L. Lions [18], [19], P. E. Souganidis [20] and G. Barles [1]. Moreover recently M. G. Crandall and P. L. Lions ([6]) proved the convergence of a general class of finite difference schemes to the viscosity solution of the model problem

$$(0.3)\qquad \begin{cases} \frac{\partial u}{\partial t} + H(Du) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N \end{cases}$$

and gave an explicit error estimate.

This paper discusses the convergence of general approximation schemes to the viscosity solution of (0.1). In particular, it contains a general theorem which roughly says that any "reasonable" scheme converges to the viscosity solution of (0.1). Under certain hypotheses explicit error estimates are also given. (Some of the arguments in the proof of

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these estimates parallel the ones in [6]). We then use this abstract theorem to establish several results concerning the convergence of Trotter products and (explicit and implicit) finite difference schemes with error estimate. Moreover, special representations (min-max) as well as the applications of the viscosity solution in differential games and control theory are discussed in connection with the above mentioned theorem.

The statement of the abstract results as they apply to (0.1) is rather lengthy and complicated. We therefore defer it to a later section. Here we describe a simple version of these results related to the model problem (0.3) and show how one can use them to obtain some representations of the viscosity solution as well as the convergence of the Trotter formulas.

To this end, for $\rho > 0$ we introduce a mapping $F(\rho) : BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$(*) such that for every $u, \hat{u} \in BUC(\mathbb{R}^N)$

(F1) $F(0)u = u$

(F2) The mapping $\rho \to F(\rho)u$ is continuous in $BUC(\mathbb{R}^N)$

(F3) There is a constant $C_1 > 0$ such that

$$\|F(\rho)u\| \le C_1\rho + \|u\| \quad (**)$$

(F4) $F(\rho)(u + k) = F(\rho)u + k$ for every $k \in \mathbb{R}$

and

(F5) $\|F(\rho)u - F(\rho)\hat{u}\| \le \|u - \hat{u}\|$

Before we state any more assumptions we should remark that, in view of a result obtained by M. G. Crandall and L. Tartar ([7]), (F4) and (F5) imply that $F(\rho)$ is order preserving in $BUC(\mathbb{R}^N)$. In particular, if for $u, v \in BUC(\mathbb{R}^N)$ it is $u(x) \le v(x)$ for every $x \in \mathbb{R}^N$, then

$$F(\rho)u(x) \le F(\rho)v(x) \quad \text{for every} \quad x \in \mathbb{R}^N .$$

(*) $BUC(\mathcal{O})$ is the Banach space of bounded real valued uniformly continuous functions defined on $\mathcal{O}$.

(**) For $u : \mathcal{O} \to \mathbb{R}$, $\|u\| = \sup_{x \in \mathcal{O}} |u(x)|$

Next we make an assumption concerning the behavior of $F(\rho)$, when it is applied to functions $u \in BUC(\mathbb{R}^N) \cap C^{0,1}(\mathbb{R}^N)^{(*)}$. In particular, we have

$$\text{(F6)} \begin{cases} \text{If } u \in C_b^{0,1}(\mathbb{R}^N)^{(**)}, \text{ then } F(\rho)u \in C_b^{0,1}(\mathbb{R}^N) \text{ and} \\ \qquad \|DF(\rho)u\| \le \|Du\|^{(***)} \\ \text{Moreover} \\ \qquad \|F(\rho)u - u\| \le C_2\rho \\ \text{where } C_2 \text{ is a constant which depends only on } \|Du\|. \end{cases}$$

Finally, we want to assume that, when applied to smooth functions, $F(\rho)$ behaves as a "generator". We have

$$\text{(F7)} \begin{cases} \text{For every } \phi \in C_b^2(\mathbb{R}^N)^{(\dagger)} \\ \qquad \left\| \frac{F(\rho)\phi - \phi}{\rho} + H(D\phi) \right\| \to 0 \\ \text{as } \rho \to 0. \text{ Moreover, for each } R > 0 \text{ the limit is} \\ \text{uniform in } \phi \text{ provided that } \|D\phi\|, \|D^2\phi\| \le R^{(\dagger\dagger)}. \end{cases}$$

Now for every partition $P = \{0 = t_0 < t_1 < \ldots < t_{n(P)} = T\}$ of $[0,T]$ and for $u_0 \in BUC(\mathbb{R}^N)$ define $u_P : \bar{Q}_T \to \mathbb{R}$ by$^{(\dagger\dagger\dagger)}$

$$\text{(0.4)} \begin{cases} u_P(x,0) = u_0(x) \\ u_P(x,\tau) = F(\tau - t_{i-1})u_P(\cdot,t_{i-1})(x) \quad \text{if } \tau \in (t_{i-1},t_i] \text{ for some} \\ i = 1,\ldots,n(P)\} . \end{cases}$$

(*),(**),(***) $C^{0,1}_{(b)}(\mathcal{O})$ is the space of (bounded) real valued Lipschitz continuous functions defined on $\mathcal{O}$. For $u \in C^{0,1}(\mathcal{O})$, $\|Du\|$ denotes the Lipschitz constant of u.

(†) $C^k_{(b)}(\mathcal{O})$ is the space of k times continuously differentiable functions defined on $\mathcal{O}$ (which together with their k derivatives are bounded).

(††) For $\phi : \mathcal{O} \to \mathbb{R}$ such that $\frac{\partial^2\phi}{\partial x_i \partial x_j}$ exist, $\|D^2\phi\| = \sum_{i,j} \left\|\frac{\partial^2\phi}{\partial x_i \partial x_j}\right\|$.

(†††) $Q_T = \mathbb{R}^N \times (0,T]$, $\bar{Q}_T = \mathbb{R}^N \times [0,T]$.

The theorem is

<u>**Theorem.**</u> Let $H : \mathbb{R}^N \to \mathbb{R}$ be continuous and assume that for every $\rho > 0$ $F(\rho) : BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ satisfies (F1), (F2), (F3), (F4), (F5), (F6) and (F7). If, for $u_0 \in BUC(\mathbb{R}^N)$ and a partition P of $[0,T]$, $u \in BUC(\bar{Q}_T)$ is the viscosity solution of (0.3) and $u_P : \bar{Q}_T \to \mathbb{R}$ is given by (0.4), then

(0.5) $$\sup_{(x,t)\in\bar{Q}_T} |u(x,t) - u_P(x,t)| \to 0 \quad \text{as} \quad |P| \to 0^{(*)}.$$

If, moreover, $H \in C^{0,1}(\mathbb{R}^N)$ and $F(\rho)$ satisfies

(F8) $$\begin{cases} \text{There is a constant } C_3 > 0 \text{ such that for every } \phi \in C_b^2(\mathbb{R}^N) \\ \left\| \dfrac{F(\rho)\phi - \phi}{\rho} + H(D\phi)\right\| \le C_3\rho(1 + \|D\phi\| + \|D^2\phi\|) \end{cases}$$

then for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$

(0.6) $$\sup_{(x,t)\in\bar{Q}_T} |u(x,t) - u_P(x,t)| \le K|P|^{1/2}$$

where K is a constant which depends only on $\|u_0\|$ and $\|Du_0\|$.

Next we describe how one can obtain, in view of the above theorem, the convergence of the Trotter products related to (0.3). In particular, for $i = 1,2$ let $H_i : \mathbb{R}^N \to \mathbb{R}$ be continuous. Then for $u_0 \in BUC(\mathbb{R}^N)$ we write $u_i(x,t) = S_i(t)u_0(x)$ for the viscosity solution of

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + H_i(Du_i) = 0 & \text{in } Q_T \\ u_i(x,0) = u_0(x) & \text{in } \mathbb{R}^N . \end{cases}$$

For $\rho > 0$ and $u \in BUC(\mathbb{R}^N)$ let $F(\rho)u : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$F(\rho)u(x) = S_2(\rho)S_1(\rho)u(x)$$

It is not hard to check (and we do so later) that $F(\rho)$ satisfies (F1), (F2), (F3), (F4), (F5) and (F6). Moreover, it is true that $F(\rho)$ satisfies (F7) (and (F8) in the case that

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(*) For a partition $P = \{0 < t_0 < t_1 < \dots < t_{n(P)} = T\}$ of $[0,T]$, $|P| = \max_i(t_i - t_{i-1})$

for $i = 1,2$ $H_i \in C^{0,1}(\mathbb{R}^N))$ with $H = H_1 + H_2$. In view of the theorem, if, for $u_0 \in BUC(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ is the viscosity solution of

$$\begin{cases} \frac{\partial u}{\partial t} + (H_1 + H_2)(Du) = 0 & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

and $P = \{0 = t_0 < t_1 < \dots < t_{n(P)} = T\}$ is a partition of $[0,T]$, then, if $t \in (t_{i-1}, t_i]$ for some $i = 1,\dots,n(P)$,

$$(0.7) \qquad u(x,t) = \lim_{|P| \to 0} (S_2(t - t_{i-1})S_1(t - t_{i-1}) \dots S_2(t_1)S_1(t_1)u_0)(x)$$

and the limit is uniform on $\bar{Q}_T$.

Finally to indicate how one goes about verifying the assumptions of the theorem we give one more example concerning a special representation of the viscosity solution. This result is closely related to the existence of the value of zero-sum differential games as we will explain later. In particular, for the problem (0.3) let us assume that $H \in C^{0,1}(\mathbb{R}^N)$ is such that

$$(0.8) \qquad H(p) = \inf_{y \in Y} \sup_{z \in Z} \{h(y,z) + f(y,z) \cdot p\}$$

where Y,Z are subsets of $\mathbb{R}^p, \mathbb{R}^q$ respectively (for some nonnegative integers p,q), $f : Y \times Z \to \mathbb{R}^N$ and $h : Y \times Z \to \mathbb{R}$ are such that

$$|h(y,z)|, \ |f(y,z)| \le B \quad \text{for} \quad (y,z) \in Y \times Z$$

and for $p,q \in \mathbb{R}^N$ $p \cdot q$ denotes the usual inner product. Then, for $\rho > 0$ and $u \in BUC(\mathbb{R}^N)$, let $F(\rho)u : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$(0.9) \qquad F(\rho)u(x) = \sup_{y \in Y} \inf_{z \in Z} \{-\rho h(y,z) + u(x - \rho f(y,z))\} .$$

It is rather trivial to check that $F(\rho) : BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ satisfies (F1), (F2), (F3), (F4), (F5) and (F6). Here we verify (F8). To this end, for $\phi \in C_b^2(\mathbb{R}^N)$ we have

$$\left|\frac{F(\rho)\phi(x)-\phi(x)}{\rho} + H(D\phi(x))\right| = \left|\sup_{y\in Y}\inf_{z\in Z}\left\{\frac{-\rho h(y,z)+\phi(x-\rho f(y,z))-\phi(x)}{\rho}\right\} + H(D\phi(x))\right| =$$

$$= \left|\sup_{y\in Y}\inf_{z\in Z}\left\{\frac{\phi(x-\rho f(y,z))-\phi(x)}{\rho} - h(y,z)\right\} + \inf_{y\in Y}\sup_{z\in Z}\{h(y,z) + f(y,z)\cdot D\phi(x)\}\right| =$$

$$= \left|\sup_{y\in Y}\inf_{z\in Z}\left\{\frac{\phi(x-\rho f(y,z))-\phi(x)}{\rho} - h(y,z)\right\} - \sup_{y\in Y}\inf_{z\in Z}\{-h(y,z) - f(y,z)\cdot D\phi(x)\}\right| \le$$

$$\le \sup_{y\in Y}\sup_{z\in Z}\left|\frac{\phi(x-\rho f(y,z))-\phi(x)}{\rho} + f(y,z)\cdot D\phi(x)\right| \le \frac{B^2}{4}\rho\|D^2\phi\| .$$

If, for $u_0 \in BUC(\mathbb{R}^N)$ and a partition P of $[0,T]$, $u \in BUC(\bar{Q}_T)$ is the viscosity solution of (0.3) with H as in (0.8) and $u_P : \bar{Q}_T \to \mathbb{R}$ is defined by (0.4), then $u_P(x,t) \to u(x,t)$ as $|P| \to 0$ and the limit is uniform on $\bar{Q}_T$. If, moreover, $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then

$$\|u - u_P\| \le K|P|^{1/2}$$

where K depends only on $\|u_0\|$ and $\|Du_0\|$. This result, in the case that

$$\sup_{y\in Y}\inf_{z\in Z}\{h(y,z) + f(y,z)\cdot p\} = \inf_{z\in Z}\sup_{y\in Y}\{h(y,z) + f(y,z)\cdot p\}$$

for every $p \in \mathbb{R}^N$, implies (because of the uniqueness of the viscosity solution) that the differential game associated with the above H has a value.

The paper is organized as follows. Section 1 recalls the definition and some properties of the viscosity solution as they are stated in [4] and [5]. Moreover, it includes the existence results and some further properties of the solution as they are stated in [1] and [20] as well as the general assumptions made on H. Section 2 is devoted to the abstract convergence theorems. In particular, two theorems are given. The first deals with schemes which satisfy an (F7) type assumption (such an assumption is identified as a "generator" property). The second theorem corresponds to schemes which do not satisfy such an assumption directly. In section 3 we obtain several min-max representations of the viscosity solution of (0.1). Section 4 contains a short discussion about two players zero-sum differential games. With the help of sections 2 and 3 it is shown here that the value of such differential games exists. Section 5 is devoted to the convergence of several

numerical schemes (explicit and fully implicit finite difference schemes) and gives error estimates. In section 6 we establish the convergence of several types of Trotter formulas. Detailed references for all the above are given in each section.

Finally we would like to thank Professor L. C. Evans for suggesting some of these problems and especially Professor M. G. Crandall for helpful discussions and good advice.

SECTION 1

We begin this section by describing the assumptions on H. Throughout this discussion we will assume:

$$(H1)\quad \begin{cases} H \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N) \text{ is uniformly continuous on} \\ [0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R) \text{ for each } R > 0^{(*)} \end{cases}$$

and

$$(H2)\quad \begin{cases} \text{There is a constant } C > 0 \text{ such that} \\ \qquad C = \sup_{(x,t)\in\bar{Q}_T} |H(t,x,0,0)| < \infty . \end{cases}$$

Moreover, we require some monotonicity of H with respect to u. More precisely, we assume:

$$(H3)\quad \begin{cases} \text{For } R > 0 \text{ there is a } \gamma_R \in \mathbb{R} \text{ such that} \\ \qquad H(t,x,r,p) - H(t,x,s,p) \geq \gamma_R(r - s) \\ \text{for } x \in \mathbb{R}^N, \ -R \leq s \leq r \leq R, \ 0 \leq t \leq T \text{ and } p \in \mathbb{R}^N . \end{cases}$$

Finally, we will have to restrict the nature of the joint continuity of H. The following Lipschitz-type assumption will be used

$$(H4)\quad \begin{cases} \text{For } R > 0 \text{ there is a constant } C_R > 0 \text{ such that} \\ \qquad |H(t,x,r,p) - H(t,y,r,p)| \leq C_R(1 + |p|)|x - y| , \\ \text{for } t \in [0,T], \ |r| \leq R \text{ and } x,y,p \in \mathbb{R}^N . \end{cases}$$

Next we state some assumptions on H which we are going to use later in addition to the above. In particular occasionally we will assume:

$$(H5)\quad \begin{cases} \text{For } R > 0 \text{ there is a } \bar{L}_R > 0 \text{ such that} \\ \qquad |H(t,x,r,p) - H(t,x,s,p)| \leq \bar{L}_R|r - s| \\ \text{for } x \in \mathbb{R}^N, \ -R \leq s \leq r \leq R, \ 0 \leq t \leq T \text{ and } p \in \mathbb{R}^N \end{cases}$$

(*) $B_N(x_0,R) = \{x \in \mathbb{R}^N : |x - x_0| \leq R\}$

$$
(H6)\begin{cases}\text{For } R > 0 \text{ there is a } N_R > 0 \\ \qquad |H(t,x,r,p) - H(\bar{t},x,r,p)| \le N_R(1 + |p|)|t - \bar{t}| \\ \text{for } t,\bar{t} \in [0,T],\ |r| \le R \text{ and } x,p \in \mathbb{R}^N \end{cases}
$$

and finally

$$
(H7)\begin{cases}\text{For } R > 0 \text{ there is a } M_R > 0 \text{ such that} \\ \qquad |H(t,x,r,p) - H(t,x,r,q)| \le M_R|p - q| \\ \text{for } t \in [0,T],\ x \in \mathbb{R}^N,\ |r| \le R \text{ and } p,q \in \mathbb{R}^N \text{ with } |p|,|q| \le R . \end{cases}
$$

We continue now with the definitions of the viscosity solution of (0.1) and (0.2). We have

Definition 1.1 ([4], [5]). Let $H \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$. A function $u \in C(Q_T)$ is a viscosity solution of

$$
\frac{\partial u}{\partial t} + H(t,x,u,Du) = 0
$$

if for every $\phi \in C^\infty(Q_T)$ (*)

(1.1) if $u - \phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then

$$
\frac{\partial \phi}{\partial t}(x_0,t_0) + H(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \le 0
$$

and

(1.2) if $u - \phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then

$$
\frac{\partial \phi}{\partial t}(x_0,t_0) + H(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \ge 0
$$

If, moreover, $u \in C(\bar{Q}_T)$ and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$, we say that u is a viscosity solution of (0.1) on $\bar{Q}_T$.

Remark 1.1. Definition 1.1 is a combination of Definition 2 and Lemma 4.1 of [4].

(*) $C^\infty_{(0)}(\ )$ is the space of infinitely many times continuously differentiable functions of compact support).

Definition 1.2 ([4], [5]). Let $H \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$, $\lambda > 0$ and $v \in C(\mathbb{R}^N)$. A function $u \in C(\mathbb{R}^N)$ is a viscosity solution of (0.2) in $\mathbb{R}^N$, if for every $\phi \in C^\infty(\mathbb{R}^N)$

(1.3) if $u - \phi$ attains a local maximum at $x_0 \in \mathbb{R}^N$, then

$$u(x_0) + \lambda H(x_0, u(x_0), D\phi(x_0)) \leq v(x_0)$$

and

(1.4) if $u - \phi$ attains a local minimum at $x_0 \in \mathbb{R}^N$, then

$$u(x_0) + \lambda H(x_0, u(x_0), D\phi(x_0)) \geq v(x_0) .$$

Next we state the theorems about the uniqueness and existence of the viscosity solution of (0.1) and (0.2) as well as some other important results of [4], [5], [1] and [20] concerning this solution.

Theorem 1.1 ([1], [20])(*). Let $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1), (H2), (H3) and (H4). For every $u_0 \in BUC(\mathbb{R}^N)$ there is a $T = T(\|u_0\|) > 0$ and $u \in BUC(\bar{Q}_T)$ such that u is the unique viscosity solution of (0.1) on $\bar{Q}_T$. If, moreover, γ_R in (H2) is independent of R, then (0.2) has a unique viscosity *solution on* $\bar{Q}_T$ *for every* $T > 0$.

Theorem 1.2 ([1], [20])(*). Let $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1), (H2), (H3) and (H4). For every $v \in BUC(\mathbb{R}^N)$ there is a $\lambda_0 = \lambda_0(\|v\|, \gamma_R)$ such that, for every $0 < \lambda < \lambda_0$, (0.2) has a unique viscosity solution $u \in BUC(\mathbb{R}^N)$ in $\mathbb{R}^N$.

Proposition 1.1 (I.3 [5], I.11 [5]). (a) Let $u \in C(Q_T)$ be a viscosity solution of

$$\frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 \quad \text{in} \quad Q_T .$$

If for $\phi \in C^1(Q_T)$ with $\phi > 0$ and $\psi \in C^1(Q_T)$

(*)The uniqueness of bounded uniformly continuous viscosity solution was proved by M. G. Crandall and P. L. Lions in [5].

(1.5) $\phi(u-\psi)$ attains a positive maximum at $(x_0,t_0) \in Q_T$, then

$$-\frac{(u(x_0,t_0)-\psi(x_0,t_0))}{\phi(x_0,t_0)}\frac{\partial\phi}{\partial t}(x_0,t_0)+\frac{\partial\psi}{\partial t}(x_0,t_0)$$

$$+H(t_0,x_0,u(x_0,t_0),-\frac{u(x_0,t_0)-\psi(x_0,t_0)}{\phi(x_0,t_0)}D\phi(x_0,t_0)+D\psi(x_0,t_0)) \le 0$$

If for $\phi \in C^1(Q_T)$ with $\phi > 0$ and $\psi \in C^1(Q_T)$

(1.6) $\phi(u-\psi)$ attains a negative minimum at $(x_0,t_0) \in Q_T$, then

$$-\frac{(u(x_0,t_0)-\psi(x_0,t_0))}{\phi(x_0,t_0)}\frac{\partial\phi}{\partial t}(x_0,t_0)+\frac{\partial\psi}{\partial t}(x_0,t_0)+$$

$$+H(t_0,x_0,u(x_0,t_0),-\frac{u(x_0,t_0)-\psi(x_0,t_0)}{\phi(x_0,t_0)}D\phi(x_0,t_0)+D\psi(x_0,t_0)) \ge 0$$

(b) Let $T > 0$, $\gamma \in \mathbb{R}$ and $g,h \in C([0,T])$. Suppose that, for every $n \in C^\infty((0,T))$, if $g-n$ attains a strict local maximum at $t_0 \in (0,T)$, we have

$$n'(t_0)+\gamma g(t_0) \le h(t_0)$$

Then for $0 \le s \le t \le T$

$$e^{\gamma t}g(t) \le e^{\gamma s}g(s)+\int_s^t e^{\gamma\tau}h(\tau)d\tau \; . \tag{1.7}$$

<u>Remark 1.2</u>. The assumptions on g in the above proposition are equivalent to saying that g is a viscosity solution of

$$g'+\gamma g \le h$$

as it is explained in [4], [5].

<u>Proposition 1.2 (VI 1. [5], IV 1. [5])</u>. (a) For $\varepsilon > 0$ let $u_\varepsilon \in C_b(\bar{Q}_T)$ be a solution of

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}-\varepsilon\Delta u_\varepsilon+H_\varepsilon(t,x,u_\varepsilon,Du_\varepsilon)=0 \quad \text{in} \quad Q_T \\ u_\varepsilon(x,0)=u_{0\varepsilon}(x) \quad \text{in} \quad \mathbb{R}^N \end{cases}$$

with $\dfrac{\partial u_\varepsilon}{\partial t}, \dfrac{\partial u_\varepsilon}{\partial x_i\partial x_j} \in C(Q_T)$. Assume $H_\varepsilon \to H$ uniformly on $[0,T]\times\mathbb{R}^N\times[-R,R]\times B_N(0,R)$

for each $R > 0$. If $\varepsilon_n \downarrow 0$ and $u_{\varepsilon_n} \to u$ locally uniformly in Q_T, then $u \in C(Q_T)$ is a viscosity solution of

$$\frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 \quad \text{in} \quad Q_T .$$

If, moreover, $u_{0\varepsilon_n} \to u_0$ uniformly on $\mathbb{R}^N$ and $u_{\varepsilon_n} \to u$ uniformly on $\bar{Q}_T$, then u is a viscosity solution of (0.1).

(b) For $\varepsilon > 0$ let $u_\varepsilon \in C^2(\mathbb{R}^N)$ be a solution of

$$-\Delta u_\varepsilon + u_\varepsilon + \lambda H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) = v_\varepsilon \quad \text{in} \quad \mathbb{R}^N .$$

Assume $H_\varepsilon \to H$ uniformly on $\mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$ and $v_\varepsilon \to v$ uniformly on $\mathbb{R}^N$. If $\varepsilon_n \downarrow 0$ and $u_{\varepsilon_n} \to u$ locally uniformly on $\mathbb{R}^N$, then $u \in C(\mathbb{R}^N)$ is a viscosity solution of (0.2).

<u>Proposition 1.3 (I.2 [5])</u>. (a) Let $u_m \in C(\bar{Q}_T)$ be a viscosity solution of

$$\begin{cases} \dfrac{\partial u_m}{\partial t} + H_m(t,x,u_m,Du_m) = 0 & \text{in} \quad Q_T \\ u_m(x,0) = u_{0m}(x) & \text{in} \quad \mathbb{R}^N \end{cases}$$

Assume $H_m \to H$ uniformly on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$. If $u_m \to u$ locally uniformly in Q_T, then u is a viscosity solution of

$$\frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 \quad \text{in} \quad Q_T$$

If, moreover, $u_{0m} \to u_0$ uniformly on $\mathbb{R}^N$ and $u_m \to u$ uniformly on $\bar{Q}_T$, then u is a viscosity solution of (0.1).

(b) Let $u_m \in C(\mathbb{R}^N)$ be a viscosity solution of $u_m + \lambda H_m(x,u_m,Du_m) = v_m$ in $\mathbb{R}^N$. Assume $H_m \to H$ uniformly on $\mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$ and $v_m \to v$ uniformly on $\mathbb{R}^N$. If $u_m \to u$ locally uniformly on $\mathbb{R}^N$, then $u \in C(\mathbb{R}^N)$ is a viscosity solution of (0.2) in $\mathbb{R}^N$.

The following results of [20] give estimates on the "off the diagonal" difference of the viscosity solutions of two problems of the form (0.1) or (0.2). Moreover, they imply

important a priori bounds on the norm, the Lipschitz constant (in the x variable) of the solution u and the differences $\|u - u_0\|$, $\|u - v\|$. To this end, let $\beta \in C_0^\infty(\mathbb{R}^N)$ be such that

$$(1.8) \qquad \begin{cases} 0 \le \beta \le 1, \quad \beta(0) = 1, \quad |D\beta| \le 2, \quad |D^2\beta| \le 4 \\ \beta(x) = 0 \quad \text{if} \quad |x| > 1 \end{cases}$$

Moreover, for $\varepsilon > 0$ let $\beta_\varepsilon(x) = \beta\left(\frac{x}{\varepsilon}\right)$. Finally, for a function $f : \mathcal{O} \to \mathbb{R}$ let

$$(1.9) \qquad \omega_f(r) = \sup_{\substack{|x-y|\le r \\ x,y\in\mathcal{O}}} |f(x) - f(y)|$$

denote the modulus of continuity of f. We have

<u>Proposition 1.4 (1.4 [20])</u>. *Let $u,\bar{u} \in BUC(\bar{Q}_T)$ be viscosity solutions of the problems*

$$\begin{cases} \frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 \quad \text{in} \quad Q_T \\ u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial \bar{u}}{\partial t} + \bar{H}(t,x,\bar{u},D\bar{u}) = 0 \quad \text{in} \quad Q_T \\ \bar{u}(x,0) = \bar{u}_0(x) \quad \text{in} \quad \mathbb{R}^N \end{cases}$$

respectively, where $u_0,\bar{u}_0 \in BUC(\mathbb{R}^N)$ and $H,\bar{H} : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1) and (H3) with the same constant $\gamma_R \le 0$ for each $R > 0$. Let $R_0 = \max(\|u\|,\|\bar{u}\|)$. If, for $R \ge R_0$, $\varepsilon > 0$ and $\gamma = \gamma_R$, $D_\varepsilon, A_\varepsilon$ are so that

$$D_\varepsilon = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \le \varepsilon\}$$

and

$$A_\varepsilon = \Big\{(t,x,y,r,p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N :$$

$$(x,y) \in D_\varepsilon, |r| \le \min\{\|u\|,\|v\|\}, |p| \le \min\Big\{\frac{6Re^{-\gamma T}}{\varepsilon} + 1, L\Big\}\Big\}$$

where

$$L = \min\{\sup_{0\le\tau\le T} \|Du(\cdot,\tau)\|, \sup_{0\le\tau\le T} \|D\bar{u}(\cdot,\tau)\|\}$$

then for every $\tau \in [0,T]$

$$(1.10)\qquad \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau) - \bar{u}(y,\tau)| + 3Re^{-\gamma\tau}\beta_\varepsilon(x-y)\} \le$$

$$\le e^{-\gamma\tau} \sup_{(x,y)\in D_\varepsilon} \{|u_0(x) - \bar{u}_0(y)| + 3R\beta_\varepsilon(x-y)\} +$$

$$+ e^{-\gamma\tau}\tau \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p) - \bar{H}(t,y,r,p)| .$$

<u>Proposition 1.5 (1.5 [20])</u>: Let $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1) and (H3) with $\gamma_R \le 0$ for every $R > 0$. If, for $u_0 \in BUC(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ is the viscosity solution of (0.1) in $\bar{Q}_T$, let $R > \|u\|$ and $\gamma = \gamma_R$. The following are true

(a) If H satisfies (H2), then for every $\tau \in [0,T]$

$$(1.11)\qquad \|u(\cdot,\tau)\| \le e^{-\gamma\tau}(\tau C + \|u_0\|)$$

where C is given by (H2)

(b) If H satisfies (H4) and $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ for every $\tau \in [0,T]$ with $L = \sup_{0\le\tau\le T} \|Du(\cdot,\tau)\|$, then for every $\tau \in [0,T]$

$$(1.12)\qquad \|Du(\cdot,\tau)\| \le e^{-\gamma\tau}(L_0 + \tau[C_R(1 + L)])$$

where $L_0 = \|Du_0\|$ and C_R is given by (H4). Moreover,

$$(1.13)\qquad L \le e^{T(2C_R e^{-\gamma T} - \gamma)}(L_0 + TC_R) .$$

(c) If $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then for every $\tau \in [0,T]$

$$(1.14)\qquad \|u(\cdot,\tau) - u_0\| \le \tau e^{-\gamma\tau} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le\|u_0\| \\ |p|\le\|Du_0\|}} |H(t,x,r,p)| .$$

(d) If, for every $\tau \in [0,T]$, $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ and $\sup_{0\le\tau\le T} \|Du(\cdot,\tau)\| \le L$, then $u \in C_b^{0,1}(\bar{Q}_T)$ and for $t,s \in [0,T]$

(1.15) $$\|u(\cdot,\tau) - u(\cdot,s)\| \le |\tau - s| e^{-\gamma T} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le\|u\| \\ |p|\le L}} |H(t,x,r,p)| .$$

Proposition 1.6 (3.3 [20]). Let $u,\bar{u} \in BUC(\mathbb{R}^N)$ be viscosity solutions of the problems

$$u + \lambda H(x,u,Du) = v \text{ in } \mathbb{R}^N \text{ and } \bar{u} + \lambda\bar{H}(x,\bar{u},D\bar{u}) = \bar{v} \text{ in } \mathbb{R}^N$$

respectively, where $H,\bar{H} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1) and (H3) with the same constant γ_R for each $R > 0$ and $v,\bar{v} \in BUC(\mathbb{R}^N)$. Let $R_0 = \max(\|u\|,\|\bar{u}\|)$. If for $R > R_0$, $\varepsilon > 0$ and $\gamma = \gamma_R$, $D_\varepsilon,\Lambda_\varepsilon$ are so that

$$D_\varepsilon = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| < \varepsilon\}$$

and

$$\Lambda_\varepsilon = \Big\{(x,y,r,p) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N : (x,y) \in D_\varepsilon,$$

$$|r| \le \min\{\|u\|,\|\bar{u}\|\}, |p| \le \min\Big\{\frac{6R}{\varepsilon} + 1, L\Big\}\Big\}$$

where

$$L = \min\{\|Du\|,\|D\bar{u}\|\}$$

and, moreover,

$$1 + \lambda\gamma > 0$$

then

(1.16) $$\sup_{(x,y)\in D_\varepsilon} \{|u(x) - \bar{u}(y)| + 3R\beta_\varepsilon(x - y)\} \le$$

$$\le \frac{1}{1 + \lambda\gamma} \sup_{(x,y)\in D_\varepsilon} \{|v(x) - \bar{v}(y)| + 3R(1 + \lambda\gamma)\beta_\varepsilon(x - y)\} +$$

$$+ \frac{\lambda}{1 + \lambda\gamma} \sup_{(x,y,s,p)\in\Lambda_\varepsilon} |H(x,s,p) - \bar{H}(y,s,p)| .$$

Proposition 1.7 (3.4 [20]). Let $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1) and (H3). If, for $v \in BUC(\mathbb{R}^N)$, $u \in BUC(\mathbb{R}^N)$ is a viscosity solution of (0.2) in $\mathbb{R}^N$, let $R > \|u\|$ and $\gamma = \gamma_R$. If $1 + \lambda\gamma > 0$, the following are true:

(a) If H satisfies (H2), then

$$\|u\| \le \frac{1}{1+\lambda\gamma}(\lambda C + \|v\|) \tag{1.17}$$

where C is given by (H2)

(b) If H satisfies (H4), $v \in C_b^{0,1}(\mathbb{R}^N)$ and $u \in C_b^{0,1}(\mathbb{R}^N)$, then

$$\|Du\| \le \frac{1}{1+\lambda\gamma}[\|Dv\| + \lambda C_R(1 + \|Du\|)] \tag{1.18}$$

where C_R is given by (H4). Moreover, if $1 + \lambda(\gamma - C_R) > 0$, then

$$\|Du\| \le \frac{1}{1+\lambda(\gamma - C_R)}(\|Dv\| + \lambda C_R) \tag{1.19}$$

(c) If $v \in C_b^{0,1}(\mathbb{R}^N)$, then

$$\|u - v\| \le \frac{\lambda}{1+\lambda\gamma} \sup_{\substack{x\in\mathbb{R}^N \\ |r|\le\|v\| \\ |p|\le\|Dv\|}} |H(x,r,p)| . \tag{1.20}$$

We conclude this section with some results concerning the behavior of the viscosity solution of (0.1) or (0.2) in the case that H satisfies (H4) and u_0 or $v \in C_b^{0,1}(\mathbb{R}^N)$ respectively. We have

<u>Proposition 1.8 (2.2 [20], 4.2 [20])</u>. (a) If, for $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying (H1), (H2), (H3) and (H4) and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ is the, provided by Theorem 1.1, viscosity solution of (0.1) in $\bar{Q}_T$, then $u \in C_b^{0,1}(\bar{Q}_T)$. Moreover, the Lipschitz constant is estimated by Proposition 1.5.

(b) If, for $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying (H1), (H2), (H3) and (H4) and $v \in C_b^{0,1}(\mathbb{R}^N)$, $u \in BUC(\mathbb{R}^N)$ is the, provided by Theorem 1.2, viscosity solution of (0.2) in $\mathbb{R}^N$, then $u \in C_b^{0,1}(\mathbb{R}^N)$. Moreover, the Lipschitz constant is estimated by Proposition 1.7.

SECTION 2

In this section we deal with the convergence of general approximation schemes to the viscosity solution of (0.1). Moreover, under certain assumptions explicit error estimates are given. In particular, we prove two theorems, which, in the applications we examine later, lead to the same conclusion, in the case that H is independent of u. The first theorem is concerned with schemes, which satisfy a generator type assumption (like (F7), (F8) in the introduction). In particular, we have

Theorem 2.1: (a) For $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying (H1), (H2) (with constant C), (H5) (with constant $\bar{L}$ independent of R) and (H4), (H6), (H7) (with constants C_R, N_R, M_R respectively for $R > 0$) and for $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ let $u \in C_b^{0,1}(\bar{Q}_T)$ be the viscosity solution of (0.1) in $\bar{Q}_T$. For $(t,\rho) \in K = \{(t,\rho) \in [0,T] \times [0,\rho_0] : 0 \le \rho \le t\}$ where $\rho_0 = \rho_0(\|u_0\|) > 0$, let $F(t,\rho,\cdot,\cdot) : C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ be such that for every $u,\bar{u},v,\bar{v} \in C_b^{0,1}(\mathbb{R}^N)$

(F1) $F(t,0,u,v) = v$.

(F2) $\begin{cases} \text{The mapping } (t,\rho) \to F(t,\rho,u,u) \text{ is} \\ \text{continuous with respect the } \|\ \| \text{ norm.} \end{cases}$

(F3) $F(t,\rho,u,v + k) = F(t,\rho,u,v) + k$ for every $k \in \mathbb{R}$.

(F4) $\|F(t,\rho,u,u) - u\| \le C_1$ where $C_1 = C_1(\|u\|,\|Du\|) > 0$.

$$\text{(F5)} \begin{cases} \text{There exists an } r > 0 \text{ and } L_1 > 0 \text{ such that if } v(x) \le \bar{v}(x) \\ \text{for every } x \in \mathbb{R}^N, \text{ then for any } y \in \mathbb{R}^N, \text{ such that} \\ \qquad |v(y + w) - v(y + \bar{w})|,\ |\bar{v}(y + w) - \bar{v}(y + \bar{w})| \le \bar{\bar{L}}|w - \bar{w}| \\ \text{for every } w,\bar{w} \in B_N(0,\rho r), \text{ it is} \\ \qquad\qquad F(t,\rho,u,v)(y) \le F(t,\rho,u,\bar{v})(y) \\ \text{where } L = \sup_{0\le\tau\le T} \|Du(\cdot,\tau)\| \text{ and } \bar{\bar{L}} = \max(L_1,L) + 1. \end{cases}$$

$$
(\mathrm{F6})\ \begin{cases} \text{There exists a constant } C_2 > 0 \text{ such that} \\ \qquad \|F(t,\rho,u,u)\| \le e^{\rho C_2}(\|u\| + \rho C_2) \\ \text{provided that } \|Du\| \le \bar{\bar{L}}\ . \end{cases}
$$

$$
(\mathrm{F7})\ \begin{cases} \text{There exist constants } C_3, C_4 > 0 \text{ such that} \\ \qquad e^{T(C_3+C_4)}(\|Du_0\| + TC_4) \le \bar{\bar{L}} \\ \text{and} \\ \qquad \|DF(t,\rho,u,u)\| \le e^{\rho(C_3+C_4)}(\|Du\| + \rho C_4) \\ \text{provided that } \|u\| \le e^{TC_2}(\|u_0\| + TC_2) \text{ and } \|Du\| \le \bar{\bar{L}} \end{cases}
$$

and

$$
(\mathrm{F8})\ \begin{cases} \text{For every } \phi \in C_b^2(\mathbb{R}^N) \text{ and } x \in \mathbb{R}^N \text{ such that } |D\phi(x)| \le L + 1, \text{ it is} \\ \qquad \left|\dfrac{F(t,\rho,u,\phi)(x) - \phi(x)}{\rho} + H(t,x,u(x),D\phi(x))\right| \le C_5(1 + \|D\phi\| + \|D^2\phi\|)\rho \\ \text{where } L = \sup\limits_{0 \le \tau \le T} \|Du(\cdot,\tau)\| \text{ and } C_5 = C_5(\|u\|,\|Du\|,L). \end{cases}
$$

For a partition $P = \{0 = t_0 < t_1 < \dots < t_{n(P)} = T\}$ of $[0,T]$, let $u_P : \bar{Q}_T \to \mathbb{R}$ be defined by

$$
(2.1)\ \begin{cases} u_P(x,0) = u_0(x) \\ u_P(x,t) = F(t, t - t_{i-1}, u_P(\cdot,t_{i-1}), u_P(\cdot,t_{i-1}))(x) \\ \text{if } t \in (t_{i-1}, t_i] \text{ for some } i = 1,\dots,n(P). \end{cases}
$$

Then there exists a constant K, which depends only on $\|u_0\|$ and $\|Du_0\|$, such that

$$
(2.2) \qquad \|u_P - u\| \le K|P|^{1/2}
$$

for $|P|$ sufficiently small.

(b) For $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying (H1), (H2), (H4) (with constant C_R for $R > 0$) and (H5) (with a constant $\bar{L}$ independent of R) and for $u_0 \in BUC(\mathbb{R}^N)$ let $u \in BUC(\bar{Q}_T)$ be the viscosity solution of (0.1) in $\bar{Q}_T$. For

$(t,\rho) \in K = \{(t,\rho) \in [0,T] \times [0,\rho_0] : 0 \le t \le \rho\}$, where $\rho_0 = \rho_0(\|u_0\|) > 0$, let $F(t,\rho,\cdot,\cdot) : BUC(\mathbb{R}^N) \times BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ be such that, for every $u,\bar{u},v,\bar{v} \in BUC(\mathbb{R}^N)$, it satisfies (F1), (F2) (for $u \in C_b^{0,1}(\mathbb{R}^N)$), (F3), (F4) (for $u \in C_b^{0,1}(\mathbb{R}^N)$) and moreover

(F9) $\begin{cases} \text{There exists a constant } C_6 > 0 \text{ such that} \\ \qquad \|F(t,\rho,u,v) - F(t,\rho,\bar{u},\bar{v})\| \le \|v - \bar{v}\| + \rho C_6 \|u - \bar{u}\| \\ \text{provided that } \bar{u},\bar{v} \in C_b^{0,1}(\mathbb{R}^N) . \end{cases}$

(F10) $\begin{cases} \text{There exists a constant } C_7 > 0 \text{ such that} \\ \qquad \|F(t,\rho,u,u)\| \le e^{\rho C_7}(\|u\| + \rho C_7). \end{cases}$

(F11) $\begin{cases} \text{If } u \in C_b^{0,1}(\mathbb{R}^N), \text{ then } F(t,\rho,u,u) \in C_b^{0,1}(\mathbb{R}^N) \text{ and} \\ \qquad \|DF(t,\rho,u,u)\| \le e^{\rho(C_8+C_9)}(\|Du\| + \rho C_9) \\ \text{where } C_8 > 0 \text{ and } C_9 = C_9(\|u\|) > 0 \end{cases}$

and

(F12) $\begin{cases} \text{For } u \in C_b^{0,1}(\mathbb{R}^N) \text{ and } \phi \in C_b^2(\mathbb{R}^N) \\ \qquad \left\|\dfrac{F(t,\rho,u,\phi) - \phi}{\rho} + H(t,\cdot,u,D\phi)\right\| \to 0 \text{ as } \rho \to 0 \\ \text{Moreover, for each } R > 0 \text{ the above limit is uniform} \\ \text{on } u,\phi, \text{ provided that } \|u\|,\|Du\|,\|D\phi\|,\|D^2\phi\| \le R. \end{cases}$

If, for a partition P of $[0,T]$, $u_P : \bar{Q}_T \to \mathbb{R}$ is defined by (2.1), then

(2.3) $$\|u_P - u\| \to 0 \quad \text{as} \quad |P| \to 0 .$$

Before we give the proof of the theorem we discuss some of its assumptions. In particular, since non-expansive mappings commuting with the addition of constants are order-preserving and vice versa (M. G. Crandall and L. Tartar [7]), (F5) is implied by (F3) and

(F13) $$\|F(t,\rho,u,v) - F(t,\rho,u,\bar{v})\| \le \|v - \bar{v}\| \quad \text{for} \quad u,v,\bar{v} \in C_b^{0,1}(\mathbb{R}^N) .$$

Similarly, (F5) together with (F3) imply that, for fixed (t,ρ) and u, $F(t,\rho,u,\cdot)$ is non-expansive on $\{u \in C_b^{0,1}(\mathbb{R}^N) : \|Du\| \le \bar{\bar{L}} + 1\}$. In several applications we are going to have (F3) and (F13), in which case the conditions on u in (F6), (F7) are irrelevant. Moreover, instead of (F8), occasionally we will assume

$$\text{(F14)} \quad \begin{cases} \text{For every } u \in C_b^{0,1}(\mathbb{R}^N) \text{ and } \phi \in C_b^2(\mathbb{R}^N) \\ \left\| \dfrac{F(t,\rho,u,\phi) - \phi) - \phi}{\rho} + H(t,\cdot,u,D\phi) \right\| \le C_{10}(1 + \|D\phi\| + \|D^2\phi\|)\rho \\ \text{where } C_{10} = C_{10}(\|u\|,\|Du\|) \end{cases}$$

which of course implies (F8). Finally, we want to remark that the important hypotheses in the theorem are the ones on F. In particular, in part (b) one can assume a more general condition than (H4) and the result is still true. However, in applications, most of the times, one needs (H4) to check (F11) and (F12). Moreover, the assumption that the constant in (H5) is independent of R has been made only for simplicity. In fact in the applications one can always reduce to this case.

Proof of Theorem 2.1. (a) We begin with a lemma, which records some of the properties of u_P. *In particular, we have:*

Lemma 2.1. For a partition $P = \{0 = t_0 < \dots < t_{n(P)} = T\}$ of $[0,T]$ and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, let $u_P : \bar{Q}_T \to \mathbb{R}$ be defined by (2.1). The following are true:

(a) For every $\tau \in [0,T]$

$$\|u_P(\cdot,\tau)\| \le e^{\tau C_2}(\tau C_2 + \|u_0\|) \tag{2.4}$$

and

$$u_P(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N) \text{ and } \|Du_P(\cdot,\tau)\| \le e^{\tau(C_3+C_4)}(\|Du_0\| + \tau C_4). \tag{2.5}$$

Moreover, if $\tau \in [t_{i-1},t_i]$ for some $i = 1,\dots,n(P)$, then

$$\|u_P(\cdot,\tau) - u_P(\cdot,t_{i-1})\| \le \bar{C}_1(\tau - t_{i-1}) \tag{2.6}$$

where $\bar{C}_1 = C_1(e^{TC_2}(\|u_0\| + TC_2), \bar{\bar{L}})$

(b) $u_P \in BUC(\bar{Q}_T)$

Proof. (a) If $\tau \in [0,t_1]$, then, since $\|Du_0\| \le \bar{\bar{L}}$,

$$\|u_P(\cdot,\tau)\| \le e^{\tau C_2}(\|u_0\| + \tau C_2)$$

and

$$\|Du_P(\cdot,\tau)\| \le e^{\tau(C_3+C_4)}(\|u_0\| + \tau C_4) \ .$$

A simple inductive argument, in view of (F6) and (F7), implies (2.4) and (2.5). Moreover, if we choose $\bar{C}_1 = C_1(e^{TC_2}(\|u_0\| + TC_2), \bar{\bar{L}})$, then (2.6) follows immediately from (2.4), (2.5) and (F4).

(b) This is obvious from (a), (F2) and the definition of u_P.

We continue now with the proof of (2.2). It is obvious that it suffices to show that there exists a constant K_1, which depends only on $\|u_0\|$ and $\|Du_0\|$, such that

$$(2.7)^{\pm} \qquad \sup_{(x,\tau)\in\bar{Q}_T} (e^{-\bar{L}\tau}(u_P(x,\tau) - u(x,\tau))^{\pm}) \le K_1|P|^{1/2} \ (*)$$

for $|P|$ sufficiently small. Here we prove only $(2.7)^+$ since the proof of (2.7) is identical. To this end, let M_P be defined by

$$M_P = \sup_{(x,\tau)\in\bar{Q}_T} [e^{-\bar{L}\tau}(u_P(x,\tau) - u(x,\tau))^+] \ .$$

Without any loss of generality we may assume

$$(2.8) \qquad M_P > 0 \ .$$

We know, in view of lemma 2.1(a), that there is an $R_1 > 0$, independent of the partition P, such that

$$\|u_P\| \le R_1 \ .$$

For $R = \max(R_1,\|u\|)$ and $\varepsilon = |P|^{1/4}$, let $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T] \to \mathbb{R}$ be defined by

---

(*) $r^+ = \max(r,0)$, $r^- = \max(-r,0)$

$$\Phi(x,y,\tau,s) = e^{-\dot{L}\frac{\tau+s}{2}}(u_p(x,\tau) - u(y,s))^+ +$$

$$+ 3(R+1)\beta_\varepsilon(x-y) + 3(R+1)\gamma_\varepsilon(\tau - s) - \frac{\tau+s}{4T} M_p$$

where $\beta_\varepsilon(\cdot) = \beta\left(\frac{\cdot}{\varepsilon}\right)$ and $\gamma_\varepsilon(\cdot) = \gamma\left(\frac{\cdot}{\varepsilon}\right)$ with

$$(2.9) \quad \begin{cases} \beta \in C_0^\infty(\mathbb{R}^N),\ \beta(0) = 1,\ 0 \le \beta \le 1,\ |D\beta| \le 2,\ |D^2\beta| \le 4,\ \beta(w) = 0 \ \text{ if } \ |w| > 1 \\ \beta(w) = 1 - |w|^2 \ \text{ for } \ |w| \le \frac{\sqrt{3}}{2} \ \text{ and} \\ \beta(w) < 1/2 \ \text{ for } \ |w| > \frac{\sqrt{3}}{2} \end{cases}$$

and

$$(2.10) \quad \begin{cases} \gamma \in C_0^\infty(\mathbb{R}),\ \gamma(0) = 1,\ 0 \le \gamma \le 1,\ |D\gamma| \le 2,\ |D^2\gamma| \le 4,\ \gamma(t) = 0 \ \text{ if } \ |t| > 1 \\ \gamma(t) = 1 - t^2 \ \text{ for } \ |t| \le \frac{\sqrt{3}}{2} \ \text{ and} \\ \gamma(t) < 1/2 \ \text{ for } \ |t| > \frac{\sqrt{3}}{2} \end{cases}$$

Since Φ is bounded on $\mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]$, for every $\delta > 0$ there is a point $(x_1,y_1,\tau_1,s_1) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]$ such that

$$\Phi(x_1,y_1,\tau_1,s_1) > \sup_{(x,y,\tau,s)\in\mathbb{R}^N\times\mathbb{R}^N\times[0,T]\times[0,T]} \Phi(x,y,\tau,s) - \delta$$

Next choose $\zeta \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ so that $0 \le \zeta \le 1$, $\zeta(x_1,y_1) = 1$, $|D\zeta| \le 1$, $|D^2\zeta| \le 1$ and define $\Psi : \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T] \to \mathbb{R}$ by

$$\Psi(x,y,\tau,s) = \Phi(x,y,\tau,s) + 2\delta\zeta(x,y)$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1,\tau_1,s_1) = \Phi(x_1,y_1,\tau_1,s_1) + 2\delta >$$

$$> \sup_{(x,y,\tau,s)\in\mathbb{R}^N\times\mathbb{R}^N\times[0,T]\times[0,T]} \Phi(x,y,\tau,s) + \delta$$

there is a point $(x_0,y_0,\tau_0,s_0) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]$ such that

$$(2.11) \quad \Psi(x_0,y_0,\tau_0,s_0) \geq \Psi(x,y,\tau,s) \quad \text{for every} \quad (x,y,\tau,s) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]$$

Moreover, for $\delta < \min\left(\frac{1}{24}, \frac{1}{8} M_p\right)$, (x_0,y_0,τ_0,s_0) has the following properties

$$(2.12) \quad \begin{cases} |x_0 - y_0| \leq \varepsilon, \ |\tau_0 - s_0| \leq \varepsilon, \ u_p(x_0,\tau_0) - u(y_0,s_0) > 0, \ \Psi(x_0,y_0,\tau_0,s_0) > 0 \\ |x_0 - y_0| \leq (L + 2\delta)\varepsilon^2 \\ \text{and} \\ \gamma_\varepsilon(\tau_0 - s_0) = 1 - \dfrac{(\tau_0 - s_0)^2}{\varepsilon^2}, \beta_\varepsilon(x_0 - y_0) = 1 - \dfrac{|x_0 - y_0|^2}{\varepsilon^2} . \end{cases}$$

To see (2.12) observe that, if for $\delta < 1/2$ either $|x_0 - y_0| > \varepsilon$ or $|\tau_0 - s_0| > \varepsilon$, then (2.11) implies

$$2(R + 1) + 3(R + 1) + 2\delta \geq \Psi(x_0,y_0,\tau_0,s_0) \geq$$

$$\geq \Psi(x,x,\tau,\tau) \geq e^{-\bar{L}\tau}(u_p(x,\tau) - u(x,\tau)) + 3(R + 1) - \frac{1}{2} M_p$$

and therefore

$$2\delta \geq \frac{1}{2} M_p + R + 1$$

which is a contradiction. Moreover, the above argument also shows that

$$\Psi(x_0,y_0,\tau_0,s_0) \geq \frac{1}{2} M_p + 6(R + 1) > 0$$

Next observe that

$$e^{-\bar{L}\frac{\tau_0+s_0}{2}}(u_p(x_0,\tau_0) - u(y_0,s_0))^+ + 2\delta \geq \frac{1}{2} M_p > 0$$

therefore for $\delta < \frac{1}{8} M_p$

$$e^{-\bar{L}\frac{\tau_0+s_0}{2}}(u_p(x_0,\tau_0) - u(y_0,s_0))^+ \geq \frac{1}{4} M_p > 0$$

i.e.

$$u_p(x_0,\tau_0) - u(y_0,s_0) > 0.$$

Moreover, (2.1) again implies that

$$2(R+1) + 3(R+1)\beta_\varepsilon(x_0 - y_0) + 3(R+1) + 2\delta \geq$$

$$\geq \Psi(x_0,y_0,\tau_0,s_0) \geq \Psi(x,x,\tau,\tau) \geq \frac{1}{2} M_p + 3(R+1) + 3(R+1)$$

therefore, since $\delta < 1/24$

$$\beta_\varepsilon(x_0 - y_0) \geq \frac{1}{3} - \frac{2\delta}{3(R+1)} \geq \frac{1}{4}$$

i.e.

(2.13)
$$\beta_\varepsilon(x_0 - y_0) = 1 - \frac{|x_0 - y_0|^2}{\varepsilon^2}$$

Similarly, we have

$$2(R+1) + 3(R+1)\gamma_\varepsilon(\tau_0 - s_0) + 3(R+1) + 2\delta \geq$$

$$\geq \Psi(x_0,y_0,\tau_0,s_0) \geq \Psi(x,x,\tau,\tau) \geq \frac{1}{2} M_p + 3(R+1) + 3(R+1)$$

and thus

(2.14)
$$\gamma_\varepsilon(\tau_0 - s_0) = 1 - \frac{|\tau_0 - s_0|^2}{\varepsilon^2}$$

Finally, observe that since

$$\Psi(x_0,y_0,\tau_0,s_0) \geq \Psi(x,y,\tau,s)$$

and

$$u_p(x_0,\tau_0) - u(y_0,s_0) > 0$$

y_0 is a minimum point for the mapping

$$y \to e^{-\frac{\bar{L}}{2}(\tau_0+s_0)} u(y,s_0) - 3(R+1)\beta_\varepsilon(x_0 - y) - 2\delta\zeta(x_0,y)$$

therefore for every $y \in \mathbb{R}^N$ it is

$$3(R+1)\beta_\varepsilon(x_0 - y) + 2\delta\zeta(x_0,y) - 3(R+1)\beta_\varepsilon(x_0 - y_0) - 2\delta\zeta(x_0,y_0) \leq$$

$$\leq e^{-\frac{\bar{L}}{2}(\tau_0+s_0)}(u(y,s_0) - u(y_0,s_0)) \leq e^{-\frac{\bar{L}}{2}(\tau_0+s_0)} L|x_0 - y_0| \leq 6(R+1)L|y - y_0|$$

But this implies that

$$|-3(R+1)D\beta_\varepsilon(x_0-y_0)+2\delta D_y\zeta(x_0,y_0)| \le 6(R+1)L$$

therefore, in view of the choice of ζ and (2.13),

$$|x_0-y_0| \le (L+2\delta)\varepsilon^2 .$$

There are several cases to be considered. These are: $\tau_0 > 0$ and $s_0 = 0$, $\tau_0 = 0$ and $s_0 > 0$ and $\tau_0 > 0$ and $s_0 > 0$. We begin with the case $\tau_0 > 0$ and $s_0 = 0$.

1st case. $\tau_0 > 0$ and $s_0 = 0$

From (2.11) and

$$\Psi(x_0,y_0,\tau_0,0) \ge \Psi(x_0,y_0,0,0)$$

in view of (2.12), it follows that

$$u_p(x_0,\tau_0) - u_0(y_0) + 3(R+1)\beta_\varepsilon(x_0-y_0) + 3(R+1)\left(1-\frac{\tau_0^2}{\varepsilon^2}\right) + 2\delta\zeta(x_0,y_0) \ge$$

$$\ge \Psi(x_0,y_0,\tau_0,0) \ge u_0(x_0) - u_0(y_0) + 3(R+1)\beta_\varepsilon(x_0-y_0) + 3(R+1) + 2\delta\zeta(x_0,y_0)$$

Therefore

$$3(R+1)|u_p(x_0,\tau_0) - u_0(x_0)| + 3(R+1)\left(1-\frac{\tau_0^2}{\varepsilon^2}\right) \ge 3(R+1)$$

But, in view of Lemma 2.1(a), we obtain:

$$\frac{\tau_0^2}{\varepsilon^2} \le \bar{C}_1\tau_0$$

and thus

(2.15) $$\tau_0 \le \bar{C}_1\varepsilon^2$$

where $\bar{C}_1$ is a constant which depends only on $\|u_0\|$ and $\|Du_0\|$ and is given by Lemma 2.1 (a). Then again (2.11) implies

$$|u_p(x_0,\tau_0) - u_0(x_0)| + |u_0(x_0) - u_0(y_0)| + 3(R+1) + 3(R+1) + 2\delta \ge$$

$$\ge \Psi(x_0,y_0,\tau_0,0) \ge \frac{1}{2}M_p + 3(R+1) + 3(R+1)$$

In view of (2.12), (2.15) and Lemma 2.1(a), we have

(2.16) $$M_p \le 2[(\bar{C}_1)^2 + L\|Du_0\|]\varepsilon^2 + 4\delta(\varepsilon^2+1)$$

2nd case. $\tau_0 = 0$ and $s_0 > 0$

From (2.11) and (2.12) it follows that

$$u_0(x_0) - u(y_0,s_0) + 3(R+1)\beta_\varepsilon(x_0 - y_0) + 3(R+1)\left(1 - \frac{s_0^2}{\varepsilon^2}\right) + 2\delta\zeta(x_0,y_0) >$$

$$> \Psi(x_0,y_0,0,s_0) > \Psi(x_0,y_0,0,0) >$$

$$> u_0(x_0) - u_0(y_0) + 3(R+1)\beta_\varepsilon(x_0 - y_0) + 3(R+1) + 2\delta\zeta(x_0,y_0)$$

therefore

$$3(R+1)|u(y_0,s_0) - u_0(y_0)| + 3(R+1)\left(1 - \frac{s_0^2}{\varepsilon^2}\right) > 3(R+1)$$

But, since $u \in C^{0,1}(\bar{Q}_T)$ with Lipschitz constant $\|Du\|$, we have

$$\frac{s_0^2}{\varepsilon^2} < \|Du\| s_0$$

and thus

(2.17) $$s_0 < \|Du\|\varepsilon^2 .$$

Then again (2.11) implies

$$|u_0(x_0) - u_0(y_0)| + |u_0(y_0) - u(y_0,s_0)| + 3(R+1) + 3(R+1) + 2\delta >$$

$$> \Psi(x_0,y_0,0,s_0) > \frac{1}{2} M_P + 3(R+1) + 3(R+1)$$

and therefore

(2.18) $$M_P < 2(\|Du\|^2 + L\|Du_0\|)\varepsilon^2 + 4\delta(\varepsilon^2 + 1) .$$

3rd case. $\tau_0 > 0$ and $s_0 > 0$

It follows for (2.11) and (2.12) that s_0 is a minimum point for the mapping $s \to e^{-\frac{L}{2}(\tau_0+s)} u(y_0,s) - 3(R+1)\gamma_\varepsilon(\tau_0 - s) + \frac{s}{4T} M_P$, therefore for $s \in [0,T]$ it is

$$3(R+1)\gamma_\varepsilon(\tau_0 - s) - \frac{s}{4T} M_p - 3(R+1)\gamma_\varepsilon(\tau_0 - s_0) + \frac{s_0}{4T} M_p <$$

$$< e^{-\frac{\bar{L}}{2}(\tau_0+s)} u(y_0,s) - e^{-\frac{\bar{L}}{2}(\tau_0+s_0)} u(y_0,s_0) <$$

$$< \left(\frac{R\bar{L}}{2} + \|Du\|\right)|s - s_0| < 6(R+1)\left(\frac{R\bar{L}}{2} + \|Du\|\right)|s - s_0|$$

But then, in view of (2.10), we have

$$6(R+1)\,\frac{|\tau_0 - s_0|}{\varepsilon^2} < 6(R+1)\left(\frac{R\bar{L}}{2} + \|Du\|\right) + \frac{1}{4T} M_p < 6(R+1)\left(\frac{R\bar{L}}{2} + \|Du\| + \frac{1}{2T}\right)$$

and thus

$$(2.19) \qquad |\tau_0 - s_0| < \left(\frac{R\bar{L}}{2} + \|Du\| + \frac{1}{2T}\right)\varepsilon^2$$

Next observe that $(x_0,\tau_0) \in Q_T$ is a point where $\phi(u_p - \psi)$ attains a positive maximum and $(y_0,s_0) \in Q_T$ is a point where $\bar{\phi}(u - \bar{\psi})$ attains negative minimum where

$$(2.20)\quad \begin{cases} \phi(x,\tau) = e^{-\frac{\bar{L}}{2}(\tau+s_0)} \\[2ex] \psi(x,\tau) = u(y_0,s_0) - 3(R+1)e^{\frac{\bar{L}}{2}(\tau+s_0)}\beta_\varepsilon(x - y_0) - 3(R+1)e^{\frac{\bar{L}}{2}(\tau+s_0)}\gamma_\varepsilon(\tau - s_0) - \\ \qquad - 2\delta e^{\frac{\bar{L}}{2}(\tau+s_0)}\zeta(x,y_0) + \dfrac{(\tau + s_0)}{4T} M_p\, e^{\frac{\bar{L}}{2}(\tau+s_0)} \\[2ex] \bar{\phi}(y,s) = e^{-\frac{\bar{L}}{2}(\tau_0+s)} \\[2ex] \bar{\psi}(y,s) = u_p(x_0,\tau_0) + 3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s)}\beta_\varepsilon(x_0 - y) + 3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s)}\gamma_\varepsilon(\tau_0 - s) + \\ \qquad + 2\delta e^{\frac{\bar{L}}{2}(\tau_0+s)}\zeta(x_0,y) - \dfrac{(\tau_0 + s)}{4T} M_p\, e^{\frac{\bar{L}}{2}(\tau_0+s)} \end{cases}$$

This observation, in view of Proposition 1.1 (a), implies that

$$0 \le \frac{\bar{L}}{2}\,(u(y_0,s_0) - \bar{\psi}(y_0,s_0)) + \frac{\bar{L}}{2}\,3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\beta_\varepsilon(x_0 - y_0) +$$

$$+ \frac{\bar{L}}{2}\,3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\gamma_\varepsilon(\tau_0 - s_0) - 3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\gamma'_\varepsilon(\tau_0 - s_0) +$$

$$+ \frac{\bar{L}}{2}\,2\delta e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\zeta(x_0,y_0) - \frac{1}{4T}\,M_P\,e^{\frac{\bar{L}}{2}(\tau_0+s_0)} +$$

$$- \frac{\bar{L}}{2}\,\frac{(\tau_0 + s_0)}{4T}\,M_P\,e^{\frac{\bar{L}}{2}(\tau_0+s_0)} + H(s_0,y_0,u(y_0,s_0),D\bar{\psi}(y_0,s_0))$$

i.e.

$$\frac{1}{4T}\,M_P e^{\frac{\bar{L}}{2}(\tau_0+s_0)} + 3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\gamma'_\varepsilon(\tau_0 - s_0) \le$$

$$\le \frac{\bar{L}}{2}\,(u(y_0,s_0) - u_P(x_0,\tau_0)) + H(s_0,y_0,u(y_0,s_0),D\bar{\psi}(y_0,s_0))$$

and, since $\tau_0 \in (t_{i-1},t_i]$ for some $i = 1,\ldots,n(P)$, also that

(2.22) $$u_P(x,t_{i-1}) \le \psi(x,t_{i-1}) + e^{-\frac{\bar{L}}{2}(\tau_0 - t_{i-1})}(u_P(x_0,\tau_0) - \psi(x_0,\tau_0)) \quad \text{for every } x \in \mathbb{R}^N$$

Moreover, in view of (2.9), (2.10), (2.11), (2.20) and the choice of ζ, the following are true:

(2.23) $$\begin{cases} |D\psi(x_0,t_{i-1}) - D\psi(x_0,\tau_0)| \le \frac{\bar{L}}{2}\,|D\psi(x_0,\tau_0)|\,|P| \\ \|D\psi(\cdot,t_{i-1})\| \le 6(R+1)e^{\bar{L}T}\left(\frac{1}{\varepsilon} + \delta\right) \\ \|D^2\psi(\cdot,t_{i-1})\| \le 12(R+1)e^{\bar{L}T}\left(\frac{1}{\varepsilon^2} + \delta\right) \end{cases}$$

Finally, during the proof of (2.12) we established that

$$|D\bar{\psi}(y_0,s_0)| = |-3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0 - y_0) + 2\delta e^{\frac{\bar{L}}{2}(\tau_0+s_0)}D_y\zeta(x_0,y_0)| \le L$$

and thus

(2.24) $$|D\psi(x_0,\tau_0)| \le L + 1/2$$

for $\delta < \frac{e^{-\bar{L}T}}{4}$, since

$$D\psi(x_0,\tau_0) = D\psi(y_0,s_0) - 2\delta e^{\frac{\bar{L}}{2}(\tau_0+s_0)}(D_y\zeta(x_0,y_0) + D_x\zeta(x_0,y_0))$$

Next observe that, if $w,\bar{w} \in B_N(0,(\tau_0 - t_{i-1})r)$ (where $r > 0$ is given by (F5)), then

$$|\psi(x_0 + w,t_{i-1}) - \psi(x_0 + \bar{w},t_{i-1})| \le$$

$$\le |\psi(x_0 + w,t_{i-1}) - \psi(x_0 + \bar{w},t_{i-1}) - D\psi(x_0,t_{i-1})\cdot(w - \bar{w})| +$$

$$+ |(D\psi(x_0,t_{i-1}) - D\psi(x_0,\tau_0))\cdot(w - \bar{w})| + |D\psi(x_0,\tau_0)\cdot(w - \bar{w})|$$

therefore, in view of (2.9), (2.23), (2.24) and the choice of ζ,

$$|\psi(x_0 + w,t_{i-1}) - \psi(x_0 + \bar{w},t_{i-1})| \le$$

$$\le \left(12(\tau_0 - t_{i-1})r\left(\frac{1}{\varepsilon^2} + \delta\right)e^{\bar{L}T} + \frac{\bar{L}}{2}(L + 1/2)|P| + L + 1/2\right)|w - \bar{w}|$$

So, since $\varepsilon = |P|^{1/4}$, if $\delta < 1$ and $|P|$ is so small that

$$|P|^{1/2}(12r(1 + |P|^{1/2})e^{\bar{L}T} + \frac{\bar{L}}{2}(L + 1/2)|P|^{1/2}) < 1/2 \tag{2.25}$$

then

$$|\psi(x_0 + w,t_{i-1}) - \psi(x_0 + \bar{w},t_{i-1})| \le \bar{\bar{L}}\,|w - \bar{w}| .$$

Moreover, and since in view of Lemma 4.1 (a)

$$|Du_p(\cdot,t_{i-1})| \le \bar{\bar{L}}$$

it is

$$|u_p(x_0 + w,t_{i-1}) - u_p(x_0 + \bar{w},t_{i-1})| \le \bar{\bar{L}}\ |w - \bar{w}|$$

and thus (F3), (F5) and (2.22) imply that

$$u_p(x_0,\tau_0) = F(\tau_0,\tau_0 - t_{i-1},u_p(\cdot,t_{i-1}),u_p(\cdot,t_{i-1}))(x_0) \le$$

$$\le F(\tau_0,\tau_0 - t_{i-1},u_p(\cdot,t_{i-1}),\psi(\cdot,t_{i-1}))(x_0) + e^{-\frac{\bar{L}}{2}(\tau_0-t_{i-1})}(u_p(x_0,\tau_0) - \psi(x_0,\tau_0))$$

Therefore we have

$$0 < \frac{F(\tau_0,\tau_0 - t_{i-1},u_P(\cdot,t_{i-1}),\psi(\cdot,t_{i-1}))(x_o) - \psi(x_0,t_{i-1})}{\tau_0 - t_{i-1}} +$$

$$+ \frac{\psi(x_0,t_{i-1}) - \psi(x_0,\tau_0)}{\tau_0 - t_{i-1}} + \frac{\left(e^{-\frac{\bar{L}}{2}(\tau_0-t_{i-1})} - 1\right)(u_P(x_0,\tau_0) - \psi(x_0,\tau_0))}{\tau_0 - t_{i-1}}$$

But since

$$\frac{e^{-\frac{\bar{L}}{2}(\tau_0-t_{i-1})} - 1}{\tau_0 - t_{i-1}} < -\frac{\bar{L}}{2} + \frac{(\bar{L})^2}{8}|P|$$

$$0 < u_P(x_0,\tau_0) - \psi(x_0,\tau_0) = e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\Psi(x_0,y_0,\tau_0,s_0) < 9(R+1)e^{\bar{L}T}$$

and

$$\frac{\psi(x_0,t_{i-1}) - \psi(x_0,\tau_0)}{\tau_0 - t_{i-1}} < -\frac{1}{4T}M_P\, e^{\frac{\bar{L}}{2}(\tau_0+s_0)} + \frac{\bar{L}}{2}(u(y_0,s_0) - \psi(x_0,\tau_0)) +$$

$$+ \bar{L}\left(\frac{R}{4T} + \frac{\bar{L}}{8}\left(6(R+1) + 1 + \frac{R}{2}\right) + 3(R+1)\right)e^{\bar{L}T}|P|$$

$$+ 3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\frac{\gamma_\varepsilon(\tau_0 - s_0) - \gamma_\varepsilon(t_{i-1} - s_0)}{\tau_0 - t_{i-1}}$$

in view of (F8), (H5), (2.23) and (2.24), the last inequality implies

$$(2.28)\quad \frac{1}{4T}M_P\, e^{\frac{\bar{L}}{2}(\tau_0+s_0)} - 3(R+1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\frac{\gamma_\varepsilon(\tau_0 - s_0) - \gamma_\varepsilon(t_{i-1} - s_0)}{\tau_0 - t_{i-1}} +$$

$$+ \frac{\bar{L}}{2}(u_P(x_0,\tau_0) - u(y_0,s_0)) < -H(\tau_0,x_0,u_P(x_0,\tau_0),D\psi(x_0,t_{i-1})) + \bar{L}\bar{C}_1|P| +$$

$$+ \bar{C}_5(1 + \|D\psi(\cdot,t_{i-1})\| + \|D^2\psi(\cdot,t_{i-1})\|)|P| + \bar{L}\left(\frac{R}{4T} + (\bar{L}+3)(R+1)\right)e^{\bar{L}T}|P|$$

where $\bar{C}_5 = C_5(R,\bar{\bar{L}})$ is given by (F8), provided that

$$(2.29)\qquad \frac{\bar{L}}{2}\left(L + \frac{1}{2}\right)|P| < \frac{1}{2}$$

Now we add (2.21) and (2.28). Using (2.20), (2.23), (2.24), (H7) with

$\bar{M} = M_{\max(R,L+1)}$ and

$$|D\psi(x_0,\tau_0) - D\bar{\psi}(y_0,s_0)| \le 4\delta e^{\bar{L}T}$$

we obtain

$$\frac{1}{2T} M_P\, e^{\frac{\bar{L}}{2}(\tau_0+s_0)} + \bar{L}(u_P(x_0,\tau_0) - u(y_0,s_0)) \le H(s_0,y_0,u(y_0,s_0)D\bar{\psi}(y_0,s_0)) -$$

$$- H(\tau_0,x_0,u_P(x_0,\tau_0),D\bar{\psi}(y_0,s_0)) + \bar{L}\Big(\frac{R}{4T} + (\bar{L} + 3)(R + 1)\Big)e^{\bar{L}T}|P| +$$

$$+ 3(R + 1)e^{\frac{\bar{L}}{2}(\tau_0+s_0)}\Big(-\gamma_\varepsilon'(\tau_0 - s_0) + \frac{\gamma_\varepsilon(\tau_0 - s_0) - \gamma_\varepsilon(t_{i-1} - s_0)}{\tau_0 - t_{i-1}}\Big) +$$

$$+ \bar{C}_5(1 + \|D\psi(\cdot,t_{i-1})\| + \|D^2\psi(\cdot,t_{i-1})\|\,|P| + \bar{L}\bar{C}_1|P| + \bar{M}(4\delta e^{\bar{L}T} + \frac{\bar{L}}{2}\Big(L + \frac{1}{2}\Big)|P|)$$

and therefore, in view of (H3), (H5), (H4), (H6), (2.12) and (2.19),

$$\frac{1}{2T} M_P\, e^{\frac{\bar{L}}{2}(\tau_0+s_0)} \le (4\delta e^{\bar{L}T} + \frac{\bar{L}}{2}\Big(L + \frac{1}{2}\Big)|P|)\bar{M} + \bar{L}\bar{C}_1|P| + \tag{2.30}$$

$$+ \bar{C}_5\Big(1 + 6(R + 1)e^{\bar{L}T}\Big(\frac{1}{\varepsilon} + \delta\Big) + 12(R + 1)e^{\bar{L}T}\Big(\frac{1}{\varepsilon^2} + \delta\Big)\Big)|P| +$$

$$+ (C_R + N_R)(1 + Le^{\bar{L}T})\Big(\frac{R\bar{L}}{2} + \frac{1}{2T} + 2\delta + L + \|Du\|\Big)\varepsilon^2$$

$$+ \frac{3(R + 1)e^{\bar{L}T}}{\varepsilon^2}|P| + \bar{L}\Big(\frac{R}{4T} + (\bar{L} + 3)(R + 1)\Big)e^{\bar{L}T}|P|$$

Combining (2.16), (2.18) and (2.30), using the fact that $\varepsilon = |P|^{1/4}$, assuming $|P| < 1$ and letting $\delta \downarrow 0$ implies that

$$M_P \le K_1|P|^{1/2}$$

and therefore

$$\sup_{(x,\tau)\in\bar{Q}_T} (u_P(x,\tau) - u(x,\tau))^+ \le K|P|^{1/2}$$

where

$$(2.31)\qquad K = e^{\bar{L}T}K_1 = 2((\bar{C}_1)^2 + L\|Du_0\| + \|Du\|^2 + L\|Du_0\|)e^{\bar{L}T} +$$

$$+ 2Te^{\bar{L}T}(\frac{\bar{L}}{2}(L + \frac{1}{2})\bar{M} + \bar{L}\bar{C}_1 + \bar{C}_5(1 + 18(R+1)e^{\bar{L}T}) +$$

$$+ (C_R + N_R)(1 + L)(\frac{R\bar{L}}{2} + \frac{1}{2T} + L + \|Du\|) + 3(R+1)e^{\bar{L}T})$$

provided that

$$(2.32)\qquad |P| < \min\left\{1, \frac{1}{\bar{L}(L + \frac{1}{2})}, \frac{1}{4}\,\frac{1}{(24re^{\bar{L}T} + \frac{\bar{L}}{2}(L + \frac{1}{2}))^2}\right\}.$$

(b) We begin with the observation that it suffices to assume $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. Indeed, if $u_0 \in BUC(\mathbb{R}^N)$, we can find a sequence $\{u_{0m}\}$ in $C_b^{0,1}(\mathbb{R}^N)$ such that $\|u_{0m} - u_0\| \to 0$ as $m \to \infty$. Then, in view of Proposition 1.4, it is

$$\|u - u_m\| \le e^{\bar{L}T}\|u_0 - u_{0m}\|$$

where u_m is the viscosity solution of (0.1) for u_{0m}. Moreover, if $u_{P,m} : \bar{Q}_T \to \mathbb{R}$ is defined by (2.1) for u_{0m}, then, if $\tau \in (t_{i-1}, t_i]$ for some $i = 1,\dots,n(P)$, we have

$$\|u_{P,m}(\cdot,\tau) - u_P(\cdot,\tau)\| \le (1 + (\tau - t_{i-1})C_6)\|u_{P,m}(\cdot,t_{i-1}) - u_P(\cdot,t_{i-1})\|$$
$$\le e^{(\tau - t_{i-1})C_6}\|u_{P,m}(\cdot,t_{i-1}) - u_P(\cdot,t_{i-1})\|$$

A simple inductive argument implies then that

$$\|u_{P,m} - u_P\| \le e^{TC_6}\|u_{0m} - u_0\|$$

Combining all the above we have

$$\|u_P - u\| \le (e^{T\bar{L}} + e^{TC_6})\|u_{0m} - u_0\| + \|u_{P,m} - u_m\|$$

For every $\alpha > 0$ there is an m such that

$$(e^{T\bar{L}} + e^{TC_6})\|u_{0m} - u_0\| < \frac{\alpha}{2}$$

But then, if $\rho_0 = \rho_0(\alpha, m) > 0$ is so that for $|P| < \rho_0$

$$\|u_{P,m} - u_m\| < \frac{\alpha}{2}$$

it is

$$\|u_P - u\| < \alpha$$

and thus the above claim is proved. For the rest we are going to assume that $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. In this case, we need the following lemma

Lemma 2.2: For a partition $P = \{0 = t_0 < \dots < t_{n(P)} = T\}$ of $[0,T]$ and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ let $u_P : \bar{Q}_T \to \mathbb{R}$ be defined by (2.1). The following are true:

(a) for every $\tau \in [0,T]$

(i) $$\|u_P(\cdot,\tau)\| \le e^{\tau C_7}(\|u_0\| + TC_7)$$

(ii) $$u_P(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N) \text{ and } \|Du_P(\cdot,\tau)\| \le e^{\tau(C_8+\bar{C}_9)}(\|Du_0\| + \tau\bar{C}_9)$$

where $\bar{C}_9 = C_9(e^{TC_7}(\|u_0\| + TC_7))$.

(iii) If $\tau \in [t_{i-1},t_i]$ for some $i = 1,\dots,n(P)$, then

$$\|u_P(\cdot,\tau) - u_P(\cdot,t_{i-1})\| \le \bar{\bar{C}}_1(\tau - t_{i-1})$$

where $\bar{\bar{C}}_1 = C_1(e^{TC_7}(\|u_0\| + TC_7),\ e^{T(C_8+\bar{C}_9)}(\|Du_0\| + T\bar{C}_9))$

(b) $u_P \in BUC(\bar{Q}_T)$

Since the lemma is proved in exactly the same way as Lemma 2.1, we omit its proof and we continue with the proof of Theorem 2.2 (b). It suffices to show that

$$\sup_{(x,\tau)\in\bar{Q}_T} \{e^{-\bar{L}\tau}(u_P(x,\tau) - u(x,\tau))^{\pm}\} \to 0 \quad \text{as} \quad |P| \to 0$$

Without any loss of generality here we prove only that if

$$M_P = \sup_{(x,\tau)\in\bar{Q}_T} (e^{-\bar{L}\tau}(u_P(x,\tau) - u(x,\tau))^{+})$$

then

$$M_P \to 0 \quad \text{as} \quad |P| \to 0 .$$

Since the proof has many similarities with that of part (a), we omit some of the details.

To this end, we claim that for every $\alpha > 0$ there is a $\rho_0 = \rho_0(\|u_0\|,\alpha) > 0$ so that if $|P| < \rho_0$, then

$$M_P < \alpha$$

thus the result. Indeed, for $\alpha > 0$ fixed but arbitrary, let $\varepsilon > 0$ be so that

$$(2.33) \qquad 2(\|Du_0\| + \|Du\| + \bar{\bar{C}}_1)\varepsilon + 2T\omega_{H,\max(\|Du\|+1,\|u\|)}(\varepsilon) < \frac{\alpha}{2}$$

where $\omega_{H,R}$ is the modulus of continuity of H on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R)$, $\bar{\bar{C}}_1$ is given by Lemma 2.2 (a) (iii) and $\|Du\| = \sup_{0<\tau<T} \|Du(\cdot,\tau)\|$. (Note that, in view of Proposition 1.8, $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ for every $\tau \in [0,T]$). For such ε choose $\rho_0 > 0$ so that if $\rho < \rho_0$, then

$$(2.34) \qquad 2T\Big[\omega_{H,\max(\|Du\|+1,\|u\|)}\Big(\frac{\bar{L}}{2}(L_1 + \frac{1}{2})\rho\Big) + \rho\bar{L}\bar{\bar{C}}_1 + \bar{L}\Big(\frac{R}{4T} + (\bar{L} + 3)(R + 1)\Big)e^{\bar{L}T}\rho +$$

$$+ \Big\|\frac{F(t,\rho,u,\phi) - \phi}{\rho} + H(t,\cdot,u,D\phi)\Big\|\Big] < \frac{\alpha}{2}$$

where $R > \max(\|u\|, e^{TC_7}(TC_7 + \|u_0\|))$ and

$$\|u\|,\|Du\|,\|D\phi\|,\|D^2\phi\| < \max\Big(R, L_1, \Big(\frac{6(R+1)}{\varepsilon} + 1\Big)e^{\bar{L}T}, \Big(\frac{12(R+1)}{\varepsilon^2} + 1\Big)e^{\bar{L}T}\Big)$$

where L_1 is given by Lemma 2.1 (a) (ii) so that

$$\|Du\|, \sup_{0<\tau<T} \|Du_P(\cdot,\tau)\| < L_1$$

and $\rho_0 > 0$ is such that

$$\frac{\bar{L}}{2}(L_1 + \frac{1}{2})\rho_0 < \frac{1}{2}\ .$$

We are going to show that if $|P| < \rho_0$, then

$$(2.35) \qquad M_P < \alpha\ .$$

To this end, let $P = \{0 = t_0 < t_1 < \dots < t_{n(P)} = T\}$ be a partition of $[0,T]$ with $|P| < \rho_0$. Without any loss of generality we assume that

$$M_P > 0$$

In this case, as in the proof of part (a), for every $\delta > 0$ we can define a continuous function $\Psi : \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T] \to \mathbb{R}$ by

$$\Psi(x,y,\tau,s) = e^{-\frac{\bar{L}}{2}(\tau + s)}(u_P(x,\tau) - u(y,s))^+ + 3(R + 1)\beta_\varepsilon(x - y) +$$

$$+ 3(R + 1)\gamma_\varepsilon(\tau - s) + 2\delta\zeta(x,y) - \frac{(\tau + s)}{4T} M_P$$

where $\zeta \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $0 \le \zeta \le 1$, $|D\zeta| \le 1$, $|D^2\zeta| \le 1$, $\beta_\varepsilon(\cdot) = \beta(\frac{\cdot}{\varepsilon})$, $\gamma_\varepsilon(\cdot) = \gamma(\frac{\cdot}{\varepsilon})$, β is as in (1.8) and γ as follows

$$(2.36)\quad \begin{cases} \gamma \in C_0^\infty(\mathbb{R}),\ 0 \le \gamma \le 1,\ \gamma(0) = 1,\ |\gamma'| \le 2,\ |\gamma''| \le 4 \\ \text{and} \\ \gamma(t) = 0 \quad \text{if} \quad |t| \ge 1 \end{cases}$$

such that there exists a point $(x_0,y_0,\tau_0,s_0) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]$ so that

$$(2.37)\quad \Psi(x_0,y_0,\tau_0,s_0) \ge \Psi(x,y,\tau,s) \quad \text{for every} \quad (x,y,\tau,s) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]$$

Moreover, as in part (a), for $\delta < \delta_0 = \min\left(\frac{1}{24}, \frac{1}{8} M_P\right)$ we have

$$(2.38)\quad \begin{cases} |x_0 - y_0| \le \varepsilon,\ |\tau_0 - s_0| \le \varepsilon,\ u_P(x_0,\tau_0) - u(y_0,s_0) > 0 \quad \text{and} \\ \Psi(x_0,y_0,\tau_0,s_0) > 0 \end{cases}$$

We have to consider the following three cases: $\tau_0 > 0$ and $s_0 = 0$, $\tau_0 = 0$ and $s_0 > 0$ and $\tau_0 > 0$ and $s_0 > 0$. We begin with the case $\tau_0 > 0$ and $s_0 = 0$

<u>1st case</u>. $\tau_0 > 0$ and $s = 0$

It follows from (2.37) that

$$e^{-\frac{\bar{L}}{2}\tau_0}(u_P(x_0,\tau_0) - u_0(y_0))^+ + 6(R + 1) + 2\delta \ge \Psi(x_0,y_0,\tau_0,0) \ge \frac{1}{2} M_P + 6(R + 1)$$

and so, in view of (2.38), for $\delta < \delta_0$

$$(2.39)\qquad M_P \le 2(\|Du_0\| + \bar{\bar{C}}_1)\varepsilon + 4\delta$$

<u>2nd case</u>. $\tau_0 = 0$ and $s_0 > 0$

Again (2.37) implies that

$$e^{-\frac{\bar{L}}{2}s_0}(u_0(x_0) - u(y_0,s_0))^+ + 6(R + 1) + 2\delta \ge \Psi(x_0,y_0,0,s_0) \ge \frac{1}{2} M_P + 6(R + 1)$$

and so, in view of (2.38), for $\delta < \delta_0$

$$(2.40)\qquad M_P \le 2\|Du\|\varepsilon + 4\delta$$

<u>3rd case</u>. $\tau_0 > 0$ and $s_0 > 0$

As in the 3rd case of the proof of part (a), it follows that for $\delta < \delta_0$, in view of (2.34), it is

$$\frac{1}{2T} M_P e^{\frac{\bar{L}}{2}(\tau_0+s_0)} \le H(s_0,y_0,u(y_0,s_0),D\bar{\psi}(y_0,s_0)) -$$

$$- H(\tau_0,x_0,u(y_0,s_0),D\bar{\psi}(y_0,s_0)) + \omega_{H,\max(R,L_1+1)}(4\delta e^{\bar{L}T} + \frac{\bar{L}}{2}(L_1 + \frac{1}{2})|P|) + \frac{\alpha}{4T}$$

where $\bar{\psi}$ is given by (2.20) and so

$$M_P \le 2T\left(\omega_{H,\max(\|u\|,\|Du\|+1)}(\varepsilon) + \omega_{H,\max(R,L_1+1)}(4\delta e^{\bar{L}T} + \frac{\bar{L}}{2}(L_1 + \frac{1}{2})|P|) + \frac{\alpha}{4T}\right) .$$

Adding (2.39), (2.40) and (2.41) and letting $\delta \downarrow 0$, in view of (2.33), we obtain (2.35).

<u>Remark 2.1</u>. It follows from the proof of Theorem 2.1 (b), that part (b) is really a result concerning Lipschitz continuous functions. In particular, if for $(t,\rho) \in K$, $F(t,\rho,\cdot,\cdot) : C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ satisfies all the assumptions of Theorem 2.1 (b) with (F9) replaced by (F13), then the conclusion of Theorem 2.1 (b) holds for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. (F9) was used only to show that if (2.2) is true for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then it is true for $u_0 \in BUC(\mathbb{R}^N)$ too.

The next theorem is concerned with schemes which, although do not satisfy a generator type assumption, can be approximated in a suitable way by schemes of type considered in Theorem 2.1. More precisely we have

<u>Theorem 2.2</u>. (a) For $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying (H1), (H2) (with a constant C), (H5) (with constant $\bar{L}$ independent of R) and (H4), (H6), (H7) (with constants C_R,N_R,M_R respectively for $R > 0$) and for $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ let $u \in C_b^{0,1}(\bar{Q}_T)$ be the viscosity solution of (0.1) in $\bar{Q}_T$. For $(t,\rho) \in K = \{(t,\rho) \in [0,T] \times [0,\rho_0] : 0 \le \rho \le t\}$, where $\rho_0 = \rho_0(\|u_0\|) > 0$, let $F(t,\rho,\cdot) : C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ be such that for every $u,\bar{u} \in C_b^{0,1}(\mathbb{R}^N)$

$$\text{(F15)} \quad \begin{cases} \text{There exists a constant } C_{10} > 0 \text{ such that} \\ \qquad \|F(t,\rho,u) - F(t,\rho,\bar{u})\| \le e^{\rho_{10}} \|u - \bar{u}\| \end{cases}$$

Moreover, suppose that for every $(t,\rho) \in K$ there exists a mapping $\bar{F}(t,\rho,\cdot,\cdot) : C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$, which satisfies the assumptions of Theorem 2.1 (a) with (F13) instead of (F5) and also

$$\text{(F16)} \quad \begin{cases} \text{For every } u \in C_b^{0,1}(\mathbb{R}^N) \\ \qquad \|F(t,\rho,u) - \bar{F}(t,\rho,u,u)\| \le C_{11}\rho^2 \\ \text{where } C_{11} = C_{11}(\|u\|,\|Du\|) \end{cases}$$

For a partition $P = \{0 = t_0 < t_1 < \dots < t_{n(P)} = T\}$ of $[0,T]$, let $u_P : \bar{Q}_T \to \mathbb{R}$ be defined by

$$\text{(2.42)} \quad \begin{cases} u_P(x,0) = u_0(x) \\ u_P(x,t) = F(t,t - t_{i-1},u_P(\cdot,t_{i-1}))(x) \text{ if } t \in (t_{i-1},t_i] \text{ for some } i = 1,\dots,n(P) \end{cases}$$

Then there exists a constant K, which depends only on $\|u_0\|$ and $\|Du_0\|$, such that

$$\text{(2.43)} \qquad \|u_P - u\| \le K|P|^{1/2}$$

for $|P|$ sufficiently small.

(b) For $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying (H1), (H2), (H4) and (H5) (with a constant $\bar{L}$ independent of R) and for $u_0 \in BUC(\mathbb{R}^N)$ let $u \in BUC(\bar{Q}_T)$ be the viscosity solution of (0.1) in $\bar{Q}_T$. For $(t,\rho) \in K = \{(t,\rho) \in [0,T] \times [0,\rho_0] : 0 \le \rho \le t\}$, where $\rho_0 = \rho_0(\|u_0\|) > 0$, let $F(t,\rho,\cdot) : BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ be such that it satisfies (F15) for every $u,\bar{u} \in BUC(\mathbb{R}^N)$. Moreover, suppose that for every $(t,\rho) \in K$ there exists a mapping $\bar{F}(t,\rho,\cdot,\cdot) : BUC(\mathbb{R}^N) \times BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$, which satisfies the assumption of Theorem 2.1 (b) and also

$$\text{(F17)} \quad \begin{cases} \text{For every } u \in C_b^{0,1}(\mathbb{R}^N) \\ \qquad \|F(t,\rho,u) - \bar{F}(t,\rho,u,u)\| = 0(\rho) \\ \text{where } 0(\rho) \text{ depends only on } \|u\| \text{ and } \|Du\| \text{ and } \dfrac{0(\rho)}{\rho} \to 0 \text{ as } \rho \to 0 \end{cases}$$

If, for a partition P of $[0,T]$, $u_P : \bar{Q}_T \to R$ is defined by (2.42), then

$$\|u_P - u\| \to 0 \quad \text{as} \quad |P| \to 0 \tag{2.44}$$

Remark 2.2. A remark analogous to the ones following Theorem 2.1 applies here too.

Proof of Theorem 2.2. (b) For $u_0, v_0 \in BUC(R^N)$ let $u_P, v_P : \bar{Q}_T \to R$ be defined by (2.42). Then by a simple inductive argument, in view of (F15), it follows that

$$\|u_P - v_P\| \le e^{TC_{10}} \|u_0 - v_0\|$$

But then it is easy to see, using the arguments at the beginning of the proof of Theorem 2.1 (b), that it suffices to prove (2.44) for $u_0 \in C_b^{0,1}(R^N)$. To this end, observe that, if $\bar{u}_P \cdot \bar{Q}_T \to R$ is defined by (2.1) for the above $\bar{F}$ and u_0, then, in view of Lemma 2.1 (a), there exist constants R_1 and L_1 which depends only on $\|u_0\|$ and $\|Du_0\|$ such that for every partition P of $[0,T]$ it is

$$\|\bar{u}_P\| \le R_1 \quad \text{and} \quad \sup_{0\le\tau\le T} \|D\bar{u}_P(\cdot,\tau)\| \le L_1$$

Next and for $\alpha > 0$ fixed but arbitrary, let $\rho_1 = \rho_1(\alpha) > 0$ be so that if $\rho \le \rho_1$ then

$$0(\rho) \le \frac{\alpha}{2Te^{C_{10}T}} \rho$$

where $0(\rho)$ is given by (F15) and corresponds to R_1 and L_1. Then, if $\tau \in [t_{i-1}, t_i]$ for some $i = 1,\dots,n(P)$, we have

$$\begin{aligned}\|u_P(\cdot,\tau) - \bar{u}_P(\cdot,\tau)\| \le \|F(\tau,\tau - t_{i-1},u_P(\cdot,t_{i-1})) - F(\tau,\tau - t_{i-1},\bar{u}_P(\cdot,t_{i-1})\| + \\ + \|F(\tau,\tau - t_{i-1},\bar{u}_P(\cdot,t_{i-1})) - \bar{F}(\tau,\tau - t_{i-1},\bar{u}_P(\cdot,t_{i-1}),\bar{u}_P(\cdot,t_{i-1}))\|\end{aligned}$$

and thus

$$\|u_P(\cdot,\tau) - \bar{u}_P(\cdot,\tau)\| \le e^{(\tau-\tau_{i-1})C_{10}}(\|u_P(\cdot,t_{i-1}) - \bar{u}_P(\cdot,t_{i-1})\| + \frac{\alpha}{2Te^{C_{10}T}}(\tau - t_{i-1})$$

An induction argument then implies that

$$\|u_p(\cdot,\tau) - \bar{u}_p(\cdot,\tau)\| < Te^{TC_{10}} \frac{\alpha}{2Te^{C_{10}T}} = \frac{\alpha}{2}$$

Finally, since by Theorem 2.1 (b) $\bar{u}_p \to u$ as $|P| \to 0$, let $\rho_2 = \rho_2(\alpha) > 0$ be such that if $\rho < \rho_0$, then

$$\|\bar{u}_p - u\| < \frac{\alpha}{2}$$

For $\rho < \min\{\rho_1,\rho_2\}$ we have

$$\|u_p - u\| < \alpha$$

which, in view of the fact that α is arbitrary, proves the result.

(a) Here because of Theorem 2.1(a), the above relations, appropriately modified so that they apply to this case, and (F16), for $|P|$ sufficiently small, we have

$$\|u_p - u\| < \bar{K}|P|^{1/2} + TC_{11}e^{TC_{10}}|P|$$

where $\bar{K}$ and C_{11} depend only on $\|u_0\|$ and $\|Du_0\|$. For $|P| < 1$ the above implies (2.43) with

$$K = \bar{K} + TC_{11}e^{TC_{10}} .$$

<u>Remark 2.3</u>. A remark analogous to Remark 2.1 applies to Theorem 2.2. In particular, if $F(t,\rho,\cdot) : C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ satisfies all the assumptions of Theorem 2.2 (b), then the results hold for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. Moreover, it follows from the proof of part (b) that we may assume $\bar{F}(t,\rho,\cdot,\cdot) : C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ satisfying all the properties. Then the result still holds.

SECTION 3

As an application of the results of Section 2 here we establish, under certain assumptions, a "min-max" representation of the viscosity solution of (0.1) in $\bar{Q}_T$. In particular, if H can be written as the max-min of certain affine functions, which satisfy suitable hypotheses, then the viscosity solution of (0.1) can be represented as the uniform limit of repeated min-max operations on the solutions of linear problems. This is important for the tneory of differential games, since it proves, as we are going to see in the next section, the existence of the "value".

To this end, let $H : [0,T] \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ be such that

$$H(t,x,u,p) = \sup_{y \in Y} \inf_{z \in Z} \{f(t,x,y,z)\cdot p + h(t,x,y,z) + g(t,x,y,z)u\} \tag{3.1}$$

where Y,Z are subsets of $\mathbf{R}^p,\mathbf{R}^q$ respectively (for p,q integers) and $f : [0,T] \times \mathbf{R}^N \times Y \times Z \to \mathbf{R}^N$, and $g,h : [0,T] \times \mathbf{R}^N \times Y \times Z \to \mathbf{R}$ satisfy the following conditions

(3.2)

If ψ is any one of three functions f,g,h, then ψ is uniformly in (t,x,y,z) *continuous in* (t,x) and satisfies a uniform Lipschitz condition in x. Let B_ψ and K_ψ be positive constants such that for every $x,\bar{x} \in \mathbf{R}^N$, $t \in [0,T]$ and $(y,z) \in Y \times Z$ it is

$$|\psi(t,x,y,z) - \psi(t,\bar{x},y,z)| \le K_\psi |x - \bar{x}|$$

and

$$|\psi(t,x,y,z)| \le B_\psi$$

It is easy to see to H satisfies (H2), (H4), (H5) and (H7) with the constant in (H5) independent of R. Indeed, in view of (3.1) and (3.2), we have

$$
(3.4)\quad \begin{cases} |H(t,x,0,0)| = |\sup_{y\in Y}\inf_{z\in Z} h(t,x,y,z)| \le B_h \\ |H(t,x,u,p) - H(t,\bar{x},u,p)| \le (K_h + K_g|u| + K_f|p|)|x-\bar{x}| \le \bar{C}(1 + |u| + |p|)|x-\bar{x}| \\ |H(t,x,u,p) - H(t,x,\bar{u},p)| \le B_g|u - \bar{u}| \\ |H(t,x,u,p) - H(t,x,u,\bar{p})| \le B_f|p - \bar{p}| . \end{cases}
$$

The above estimates, together with the uniform in (t,x,y,z) continuity of f,g,h in (t,x), imply that H also satisfies (H1). But then, in view of Theorem 1.1, for every $u_0 \in BUC(\mathbb{R}^N)$ the problem (0.1) for the above H has a unique viscosity solution $u \in BUC(\bar{Q}_T)$ in $\bar{Q}_T$. Moreover, in view of (1.2), for every $\tau \in [0,T]$ we have

$$
(3.5)\qquad \|u(\cdot,\tau)\| \le e^{\tau B_g}(\tau B_h + \|u_0\|)
$$

Next, for $(t,\rho) \in K = \{(t,\rho) \in [0,T] \times [0,T] : 0 \le \rho \le t\}$ and $u,v \in BUC(\mathbb{R}^N)$ let $F(t,\rho,u,v) : \mathbb{R}^N \to \mathbb{R}$ and $\bar{F}(t,\rho,u) : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$
(3.6)\qquad F(t,\rho,u,v)(x) = \inf_{y\in Y}\sup_{z\in Z} \{-\rho h(t,x,y,z) - \rho g(t,x,y,z)u(x) + v(x - \rho f(t,x,y,z))\}
$$

and

$$
(3.7)\qquad \bar{F}(t,\rho,u)(x) = \inf_{y\in Y}\sup_{z\in Z} \{-\rho h(t,x,y,z) + e^{-\rho g(t,x,y,z)}u(x - \rho f(t,x,y,z))\}
$$

The theorem is

<u>Theorem 3.1</u>. (a) For $u_0 \in BUC(\mathbb{R}^N)$ and a partition P of $[0,T]$ let $u_P, \bar{u}_P : \bar{Q}_T \to \mathbb{R}$ be defined by (2.1), (2.38) respectively using the above F and $\bar{F}$. If $u \in BUC(\bar{Q}_T)$ is the viscosity solution of (0.1) in $\bar{Q}_T$ for the above u_0 and H, then

$$
(3.8)\qquad \|u_P - u\| \to 0 \quad \text{as} \quad |P| \to 0
$$

and

$$
(3.9)\qquad \|\bar{u}_P - u\| \to 0 \quad \text{as} \quad |P| \to 0
$$

(b) If, moreover, $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ and ψ in (3.2) also satisfies

$$
(3.10)\qquad |\psi(t,x,y,z) - \psi(\bar{t},x,y,z)| \le K_\psi|t - \bar{t}|
$$

then, for $|P|$ sufficiently small,

(3.11) $$\|u_P - u\| \leq K|P|^{1/2}$$

and

(3.12) $$\|\bar{u}_P - u\| \leq K|P|^{1/2}$$

where K is a constant which depends only on $\|u_0\|$ and $\|Du_0\|$.

Remark 3.1. The convergence of $\bar{u}_P$ as $|P| \to 0$ (in the local uniform topology) was proved by W. H. Fleming in [13] via considerations of stochastic differential games. The limit function obtained in [13], in view of Proposition 1.2, is the viscosity solution of (0.1) in $\bar{Q}_T$.

Proof of Theorem 3.1. It suffices to check the assumptions of Theorem 2.1 and 2.2. We begin by proving (3.8) and (3.11). In view of (3.2), it is obvious that $F(t,\rho,u,v) \in BUC(\mathbb{R}^N)$. Moreover

$$F(t,0,u,v)(x) = \inf_{y \in Y} \sup_{z \in Z} v(x) = v(x)$$

and thus (F1). To verify (F2) observe that for $u,v \in BUC(\mathbb{R}^N)$ and $(t,\rho),(\bar{t},\bar{\rho}) \in K$ we have

$$\|F(t,\rho,u,v) - F(\bar{t},\bar{\rho},u,v)\| \leq \omega_v(|\rho - \bar{\rho}|B_f + \bar{\rho}\omega_f(|t - \bar{t}|)) +$$
$$+ |\rho - \bar{\rho}|(B_h + B_g\|u\|) + \bar{\rho}(\omega_h(|t - \bar{t}|) + \|u\|\omega_g(|t - \bar{t}|))$$

and therefore the result. Next, for $u,v,\bar{u},v \in BUC(\mathbb{R}^N)$ and $k \in \mathbb{R}$, in view of (3.6), it is

$$|F(t,\rho,u,v)(x)| \leq \sup_{y \in Y} \sup_{z \in Z} (\rho B_h + \rho B_g\|u\| + \|v\|) \leq \|v\| + \rho(B_g + B_h)(1 + \|u\|)$$

$$\begin{aligned} F(t,\rho,u,v+k)(x) &= \inf_{y \in Y} \sup_{z \in Z}\{-\rho h(t,x,y,z) - \rho g(t,x,y,z)u(x) + v(x-\rho f(t,x,y,z)) + k\} \\ &= \inf_{y \in Y} \sup_{z \in Z}\{-\rho h(t,x,y,z) - \rho g(t,x,y,z)u(x) + v(x-\rho f(t,x,y,z))\} + k \\ &= F(t,\rho,u,v)(x) + k \end{aligned}$$

and

$$|F(t,\rho,u,v)(x) - F(t,\rho,\bar{u},\bar{v})(x)| \le \|v - \bar{v}\| + \rho B_g\|u - \bar{u}\|$$

thus (F3), (F9) and (F10) hold.

Now assume that $u,v \in C_b^{0,1}(\mathbb{R}^N)$. For $(t,\rho) \in K$ and $x,\bar{x} \in \mathbb{R}^N$ we have

$$|F(t,\rho,u,v)(x) - F(t,\rho,u,v)(\bar{x})| \le [\rho(K_h + K_g\|u\| + B_g\|Du\|) + \|Dv\|(1 + \rho K_f)]|x-\bar{x}|$$

therefore $F(t,\rho,u,v) \in C_b^{0,1}(\mathbb{R}^N)$ and

$$\|DF(t,\rho,u,v)\| \le e^{\rho K_f}(\|Dv\| + \rho(K_h + K_g\|u\| + B_g)(1 + \|Du\|))$$

Moreover

$$|F(t,\rho,u,u)(x) - u(x)| \le \rho(B_h + B_g\|u\| + B_f\|Du\|)$$

and thus (F4) and (F11). Finally, for $u \in C_b^{0,1}(\mathbb{R}^N)$, $\phi \in C_b^2(\mathbb{R}^N)$, $t \in [0,T]$, $x \in \mathbb{R}^N$ and $\rho > 0$, in view of (3.1) and (3.6), we have

$$\left|\frac{F(t,\rho,u,\phi)(x) - \phi(x)}{\rho} + H(t,x,u(x),D\phi(x))\right| =$$

$$= \Big|\sup_{y\in Y}\inf_{z\in Z}\left(\frac{-\rho h(t,x,y,z) - \rho g(t,x,y,z)u(x) + \phi(x - \rho f(t,x,y,z)) - \phi(x)}{\rho}\right) +$$

$$+ \inf_{y\in Y}\sup_{z\in Z}\,(h(t,x,y,z) + g(t,x,y,z)u(x) + f(t,x,y,z)\cdot D\phi(x))\Big| +$$

$$= \Big|\sup_{y\in Y}\inf_{z\in Z}\left(-h(t,x,y,z) - g(t,x,y,z)u(x) + \frac{\phi(x - \rho f(t,x,y,z)) - \phi(x)}{\rho}\right) -$$

$$- \sup_{y\in Y}\inf_{z\in Z}\,(-h(t,x,y,z) - g(t,x,y,z)u(x) - f(t,x,y,z)\cdot D\phi(x))\Big| \le$$

$$\le \sup_{y\in Y}\sup_{z\in Z}\left|\frac{\phi(x - \rho f(t,x,y,z)) - \phi(x)}{\rho} + f(t,x,y,z)\cdot D\phi(x)\right| \le \frac{\rho}{2}B_f^2\|D^2\phi\|$$

and therefore (F14).

Then Theorem 2.1 (b) implies (3.8) and Theorem 2.1 (a) implies (3.9) since, in view of (3.10), we have

$$|H(t,x,u,p) - H(\bar{t},x,u,p)| \le (K_h + K_g|u| + K_f|p|)|t - \bar{t}|$$

i.e. H satisfies (H6).

For the proof of (3.9) and (3.11) it suffices to observe that for $u \in C_b^{0,1}(\mathbb{R}^N)$

$$|\bar{F}(t,\rho,u)(x) - F(t,\rho,u,u)(x)| \le \sup_{y\in Y}\sup_{z\in Z} |e^{-\rho g(t,x,y,z)}u(x - \rho f(t,x,y,z)) +$$

$$+ \rho g(t,x,y,z)u(x) - u(x - \rho f(t,x,y,z))| \le \rho^2(B_g B_f \|Du\| + \frac{1}{2} B_g^2 \|u\| e^{TB_g}) .$$

Indeed, then we can use $F(t,\rho,\cdot,\cdot)$ for $\bar{F}(t,\rho,\cdot,\cdot)$ in Theorem 2.2 and the result follows immediately, since for $u,\bar{u} \in BUC(\mathbb{R}^N)$, in view of (3.7), it is

$$\|\bar{F}(t,\rho,u) - \bar{F}(t,\rho,\bar{u})\| \le e^{\rho B_g}\|u - \bar{u}\|$$

and thus (F15).

<u>Remark 3.2</u>. Obviously an analogous theorem can be proved in the case that

$$H(t,x,u,p) = \inf_{y\in Y}\sup_{z\in Z} \{f(t,x,y,z)\cdot p + h(t,x,y,z) + g(t,x,y,z)u\} \tag{3.13}$$

and

$$\left\{\begin{aligned} F(t,\rho,u,v)(x) &= \sup_{y\in Y}\inf_{z\in Z} \{-\rho h(t,x,y,z) - \rho g(t,x,y,z)u(x) + v(x - \rho f(t,x,y,z))\} \\ \bar{F}(t,\rho,u)(x) &= \sup_{y\in Y}\inf_{z\in Z} \{-\rho h(t,x,y,z) + e^{-\rho g(t,x,y,z)}u(x - \rho f(t,x,y,z))\} \end{aligned}\right. \tag{3.14}$$

where f,g,h satisfy (3.2). Moreover, either Y or Z can be empty. For example, if $Y = \emptyset$ and

$$H(t,x,u,p) = \inf_{z\in Z} \{f(t,x,z)\cdot p + h(t,x,z) + g(t,x,z)u\} \tag{3.15}$$

then the results of Theorem 3.1 apply to this case too.

<u>Remark 3.3</u>. L. C. Evans proved in [9] a different min-max representation for the viscosity solution of (0.3) under the assumption that $H \in C^1(\mathbb{R}^N)$ $C^{0,1}(\mathbb{R}^N)$ and $u_0 \in BUC(\mathbb{R}^N)$ with $\lim_{|x|\to\infty} |u(x)| = 0$.

<u>Remark 3.4</u>. For examples of functions H which can be put in the form (3.1) we refer the reader to L. C. Evans [9], L. C. Evans and P. E. Souganidis [10] and W. H. Fleming [13].

SECTION 4.

We begin with a short discussion about two-person zero sum fixed duration differential games. To this end, consider a system of N-differential equations

$$\frac{d\xi}{ds} = f(s,\xi,y,z) \qquad (s_0 < s < T) \tag{4.1}$$

with initial condition

$$\xi(s_0) = \xi_0 \in R^N \tag{4.2}$$

where Y,Z are compact subsets of some R^p and R^q respectively and $f : [s_0,T] \times R^N \times Y \times Z \to R^N$ satisfies certain assumptions, which we are going to specify later.

The set $Y(Z)$ is called the control set for the player $I(II)$. A measurable function $y(s)$ $(z(s))$ with values in $Y(Z)$ for almost every t is called a control function for $I(II)$. If we substitute into (4.1) any control functions $y = y(s)$, $z(s)$ $(s_0 < s < T_0)$, then we obtain a system

$$\frac{d\xi}{ds} = f(s,\xi(s),y(s),z(s)) \qquad (s_0 < s < T) \tag{4.3}$$

The conditions on f are going to be such that (4.3), (4.2) has a unique solution $\xi : [s_0,T] \to R^N$, i.e. ξ is the unique absolutely continuous function for which

$$\xi(s) = \xi_0 + \int_{s_0}^{T} f(\rho,\xi(\rho),y(\rho),z(\rho))d\rho$$

for every $s \in [s_0,T]$. We call $\xi(s)$ the trajectory corresponding to $y(s)$, $z(s)$.

Given any control functions $y(s)$, $z(s)$ for $s_0 < s < T$, let $\xi(s)$ be the corresponding trajectory. We introduce the functional

$$P(y,z) = e^{\int_{s_0}^{T} g(s,\xi(s),y(s),z(s))ds} \; u_0(\xi(T)) + \int_{s_0}^{T} h(s,\xi(s),y(s),z(s))ds \tag{4.4}$$

where $u_0 : R^N \to R$ and $g,h : [s_0,T] \times R^N \times Y \times Z \to R$ are some given functions. P is called the <u>payoff functional</u>. In view of the previous remarks, the payoff $P(y,z)$ is well defined for each choice of controls made by the players I and II.

Finally, we assume that the earlier introduced functions f, g and h are defined on $[0,T] \times \mathbb{R}^N \times Y \times Z$ and we define $H^{\pm} : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$
(4.5) \quad \begin{cases} H^{+}(t,x,u,p) = \inf_{z \in Z} \sup_{y \in Y} \{f(t,x,y,z) \cdot p + g(t,x,y,z)u + h(t,x,y,z)\} \\ \text{and} \\ H^{-}(t,x,u,p) = \sup_{y \in Y} \inf_{z \in Z} \{f(t,x,y,z) \cdot p + g(t,x,y,z)u + h(t,x,y,z)\} \end{cases}
$$

We say that the minimax condition (or Isaacs condition) is satisfied if

(4.6) $\quad H^{+}(t,x,u,p) = H^{-}(t,x,u,p)$ for every $t \in [0,T]$, $x \in \mathbb{R}^N$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^N$

When this condition holds, we set $H(t,x,u,p) = H^{+}(t,x,u,p) = H^{-}(t,x,u,p)$ and call this function the Hamiltonian function of the differential game.

A two-person zero sum differential game of fixed duration consists of the system of differential equations (4.3), (4.1) and the payoff functional (4.4). It is a zero sum game, because the aim of the player I is to maximize the payoff and the aim of the player II is to minimize it. Throughout the differential game and at any time t each player has complete information about the past (i.e. he knows everything that his opponent did), but he has no information about the present and the future choices of controls by his opponent. The value of the differential game described above should be the value of the payoff when both players use their optimal strategies, which, however, do not exist in general. This leads to several alternative definitions of the value as we are going to explain shortly.

Differential games were first studied by Isaacs ([17]). One of his main contributions was the heuristic derivation of the fact that the value of the game should satisfy a Hamilton-Jaccbi type equation, in particular the Isaacs-Bellman equation. Later W. H. Fleming ([11], [12], [13], [14]) studied differential games by discretizing time and solving difference equations instead of (4.3), (4.2). He defined an upper and lower value depending on whether the I or the II player moves first at each step (i.e. has the advantage) and he examined whether these values exist and if yes whether they are equal. Fleming introduced "noise" into the game and so into the approximating discrete difference games. He was then able to show that the upper and lower values of the approximating games

converge as the amount of noise decreased to zero. After that A. Friedman ([15], [16]), and later R. J. Elliott and N. J. Kalton ([8]) studied differential games directly, i.e. by not approximating by difference equation. However, Friedman introduced the idea of upper and lower strategy varying at only finitely many division points. Moreover, he defined an upper and a lower value depending on which player chooses his control first at each division point. Then Friedman ([15], [16]) and Elliott and Kalton ([8]) again introduced "noise" and were able to show that the upper and lower values of the approximating games (stochastic differential games) exist. All the above (Fleming, Friedman, Elliott and Kalton) defined as the value of the game the common value of the upper and lower value in the case they coincide. Moreover, they proved that both the upper and the lower value satisfy, under certain conditions, in the almost everywhere sense appropriate Hamilton-Jacobi type equations, which are, if the Isaacs condition holds, the Isaacs-Bellman equation. Using Proposition 1.2 and the fact that the upper and lower value of the approximating problems (when noise is introduced) satisfy parabolic equations of the type in Proposition 1.2 and moreover converge, as the amount of noise goes to zero, M. G. Crandall and P. L. Lions [5] obtained that, under certain assumptions, the upper and lower value are viscosity solutions of Hamilton-Jacobi type equations. Thus in the case that the Isaacs condition is satisfied, the differential game has a value by the uniqueness of the viscosity solution.

Here we use the results of Sections 2 and 3 to show directly (i.e. without introducing noise) that the upper and lower value exist. Moreover, if the Isaacs condition holds, then the value in the sense of either Fleming or Friedman exists. It is an immediate of the proof then that these two notions (when comparable) coincide.

(a) <u>The value in the sense of Fleming</u>

We begin with the following assumptions on f, h, g, u_0

$$
\text{(FL1)} \quad \begin{cases} f \in C([0,T] \times \mathbb{R}^N \times Y \times Z). \text{ Moreover, there exists a constant } K_f \text{ such} \\ \text{that for every } t \in [0,T], \ y \in Y, \ z \in Z \text{ and } x, \bar{x} \in \mathbb{R}^N \text{ it is} \\ \qquad |f(t,x,y,z) - f(t,\bar{x},y,z)| \le K_f |x - \bar{x}| \end{cases}
$$

(FL2) $h \in C([0,T] \times \mathbb{R}^N \times Y \times Z)$, $g \in C([0,T] \times \mathbb{R}^N \times Y \times Z))$

(FL3) $u_0 \in C(\mathbb{R}^N)$

Next, for a partition $P = \{0 = t_0 < t_1 < \dots < t_{n(P)} = T\}$ let $W_P^+, W_P^- : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ be defined by

$$(4.7)\quad \begin{cases} W_P^+(x,T) = u_0(x) \\ W_P^+(x,t) = \inf\limits_{z\in Z}\sup\limits_{y\in Y}\{(t_{j+1} - t)h(t,x,y,z) + \\ \qquad + e^{(t_{j+1}-t)g(t,x,y,z)} W_P^+(x + (t_{j+1} - t)f(t,x,y,z),t_{j+1})\} \\ \text{if } t \in [t_j,t_{j+1}) \text{ for } j = 0,\dots,n(P) - 1 \end{cases}$$

and

$$(4.8)\quad \begin{cases} W_P^-(x,T) = u_0(x) \\ W_P^-(x,t) = \sup\limits_{y\in Y}\inf\limits_{z\in Z}\{(t_{j+1} - t)h(t,x,y,z) \\ \qquad + e^{(t_{j+1}-t)g(t,x,y,z)} W_P^-(x + (t_{j+1} - t)f(t,x,y,z),t_{j+1})\} \\ \text{if } t \in [t_j,t_{j+1}) \text{ for } j = 0,\dots,n(P) - 1 \end{cases}$$

For $(x,t) \in \mathbb{R}^N \times [0,T]$, $W_P^+(W_P^-)$ corresponds to an approximation of the differential game with dynamics (4.1), initial condition (x,t) and payoff (4.4). The question is whether $\lim\limits_{|P|\to 0} W_P^+(x,t)$ and $\lim\limits_{|P|\to 0} W_P^-(x,t)$ exist and if yes whether the limits are equal. We need the following definition.

<u>Definition 4.1</u> (W. H. Fleming [11], [12]). For a partition P of $[0,T]$ and $(x,t) \in \bar{Q}_T$, let $W_P^+(x,t)$ and $W_P^-(x,t)$ be defined by (4.7) and (4.8) respectively. If $\lim\limits_{|P|\to 0} W_P^+(x,t) = W^+(x,t)$ $\left(\lim\limits_{|P|\to 0} W_P^-(x,t) = W^-(x,t)\right)$ exists, then $W^+(x,t)(W^-(x,t))$ is the upper (lower) value of the differential game with dynamics (4.1), initial condition (x,t) and payoff (4.4). If, moreover, $W^+(x,t) = W^-(x,t)$, then $W(x,t) = W^+(x,t) = W^-(x,t)$ is the value of this differential game.

Now we state and prove the theorem, which establishes the existence of the upper and lower value and, under the Isaacs condition, the existence of the value of a differential game. We have

Theorem 4.1. If (FL1), (FL2) and (FL3) are satisfied, then, for every $(x,t) \in \bar{Q}_T$, the upper (lower) value of the differential game with dynamics (4.1), initial condition (x,t) and payoff (4.4) exists. Moreover, if the Isaacs condition holds, then the value of this differential game exists. Finally, if f,h,g and u_0 also satisfy

$$\text{(FL4)}\quad \begin{cases} \text{If } \psi \text{ is any one of the functions } f,h \text{ and } g, \text{ then} \\ \text{there exists a constant } K_\psi \text{ such that for every} \\ t,\bar{t} \in [0,T],\ x,\bar{x} \in \mathbb{R}^N,\ y \in Y \text{ and } z \in Z \text{ it is} \\ \qquad |\psi(t,x,y,z) - \psi(\bar{t},\bar{x},y,z)| \le K_\psi(|t-\bar{t}| + |x-\bar{x}|) \end{cases}$$

and

(FL5) $\quad u_0 \in C_b^{0,1}(\mathbb{R}^N)$

then, for every $(x,t) \in \bar{Q}_T$ and $|P|$ sufficiently small, it is

$$|W_P^{\pm}(x,t) - W^{\pm}(x,t)| \le K|P|^{1/2} \tag{4.9}$$

where K is a constant which depends only on $\|u_0\|$, $\|Du_0\|$ and $\bar{R} > |x|$.

Remark 4.1. Conditions (FL1), (FL2) and (FL3) are more general than (FL4) and (FL5), which are used in [11], [12], [13]. Moreover, here we use partitions of arbitrary mesh. Finally estimates of the form (4.9) seem to be new in this context.

Proof of Theorem 4.1. Let $(x,t) \in \bar{Q}_T$ be fixed and choose $\bar{R} > 0$ so that

$$|x| \le \bar{R}\ .$$

If $P = \{0 = t_0 < t_1 < \dots < t_{n(P)} = T\}$ is a partition of $[0,T]$, let i be such that

$$t \in [t_i, t_{i+1})$$

Then, for $(y,z) \in Y \times Z$, consider the sequence $\{x_j(y,z)\}_{j=i,\dots,n(P)}$ in $\mathbb{R}^N$ defined by

$$\text{(4.10)}\quad \begin{cases} x_i(y,z) = x \\ x_{i+1}(y,z) = x_i(y,z) + (t_{i+1} - t)f(t,x_i(y,z),y,z) \\ x_j(y,z) = x_{j-1}(y,z) + (t_j - t_{j-1})f(t_{j-1},x_{j-1}(y,z),y,z) \quad \text{for } j = i+2,\dots,n(P) \end{cases}$$

A simple induction argument then implies that, for any $j = i,\ldots,n(P)$, it is

$$|x_j| \leq R$$

where

$$R = e^{K_f T}(\bar{R} + TB_{1,f})$$

with

(4.12) $$B_{1,f} = \sup_{(t,y,z)\in[0,T]\times Y\times Z} |f(t,0,y,z)|$$

We continue by truncating f, h, g, u_0, H^+ and H^-. In particular, we define $f_R : [0,T] \times \mathbf{R}^N \times Y \times Z \to \mathbf{R}^N$, $g_R : [0,T] \times \mathbf{R}^N \times Y \times Z \to \mathbf{R}$, $h_R : [0,T] \times \mathbf{R}^N \times Y \times Z \to \mathbf{R}$, $u_{0,R} : \mathbf{R}^N \to \mathbf{R}$ and H_R^+, $H_R^- : [0,T] \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ by

$$f_R(\tau,w,y,z) = \begin{cases} f(\tau,w,y,z) & \text{if } |w| \leq R \\ f(\tau, \frac{w}{|w|}R,y,z) & \text{if } |w| > R \end{cases}$$

$$g_R(\tau,w,y,z) = \begin{cases} g(\tau,w,y,z) & \text{if } |w| \leq R \\ g(\tau, \frac{w}{|w|}R,y,z) & \text{if } |w| > R \end{cases}$$

$$h_R(\tau,w,y,z) = \begin{cases} h(\tau,w,y,z) & \text{if } |w| \leq R \\ h(\tau, \frac{w}{|w|}R,y,z) & \text{if } |w| > R \end{cases}$$

$$u_{0,R}(w) = \begin{cases} u_0(w) & \text{if } |w| \leq R \\ u_0\left(\frac{w}{|w|}R\right) & \text{if } |w| > R \end{cases}$$

$$H_R^+(\tau,w,r,p) = \inf_{z\in Z}\sup_{y\in Y}\{f_R(\tau,w,y,z)\cdot p + g_R(\tau,w,y,z)r + h_R(\tau,w,y,z)\}$$

and

$$H_R^-(\tau,w,r,p) = \sup_{y\in Y}\inf_{z\in Z}\{f_R(\tau,w,y,z)\cdot p + g_R(\tau,w,y,z)r + h_R(\tau,w,y,z)\}$$

It is easy to check that

(i) f_R, g_R, h_R and $u_{0,R}$ are bounded uniformly continuous functions

(ii) f_R is uniformly in (τ,w,y,z) Lipschitz continuous in w with Lipschitz constant K_f.

(iii) If the Isaacs condition holds for H^+ and H^-, then it holds for H_R^+ and H_R^- too.

Next and for each positive integer n choose $h^n, g^n \in BUC([0,T] \times \mathbb{R}^N \times Y \times Z)$ and $u_0^n \in BUC(\mathbb{R}^N)$, so that they satisfy (FL4) and (FL5) and moreover

$$\|u_0^n\| \le \|u_0\|$$

and

$$(4.13)\quad \begin{cases} \text{For every } \tau \in [0,T],\ w \in \mathbb{R}^N,\ y \in Y \text{ and } z \in Z \\ |h^n(\tau,w,y,z) - h_R(\tau,w,y,z)| \le \frac{1}{n} \\ |g^n(\tau,w,y,z) - g_R(\tau,w,y,z)| \le \frac{1}{n} \\ |u_0^n(w) - u_{0,R}(w)| \le \frac{1}{n} \end{cases}$$

For each n and any partition P of $[0,T]$, define $W_{P,R}^{+,n} : \bar{Q}_T \to \mathbb{R}$ and $W_{P,R}^{-,n} : \bar{Q}_T \to \mathbb{R}$ by (4.7) and (4.8) using f_R, h^n, g^n and $u_{0,R}$. If, for the given $(x,t) \in \bar{Q}_T$ and any $(y,z) \in Y \times Z$, we define $\bar{x}_j(y,z)$ by (4.10) using f_R, then, since

$$\sup_{(\tau,y,z)\in[0,T]\times Y\times Z} |f_R(\tau,0,y,z)| = \sup_{(\tau,y,z)\in[0,T]\times Y\times Z} |f(\tau,0,y,z)|$$

we have that, for every n, it is

$$(4.14)\qquad |\bar{x}_j(y,z)| \le R$$

But then this, in view of (4.7), (4.8) and (4.13), implies that

$$(4.15)\qquad |W_P^{\pm}(x,t) - W_{P,R}^{\pm,n}(x,t)| \le \frac{1}{n} e^{TB_R}[T(W_R + 1) + 1]$$

where

$$B_R = \sup_{(\tau,x,y,z)\in[0,T]\times\mathbb{R}^N\times Y\times Z} |g_R(\tau,w,y,z)|$$

and

$$W_R = e^{TB_R}\Big(\|u_{0,R}\| + T \sup_{(\tau,x,y,z)\in[0,T]\times\mathbb{R}^N\times Y\times Z} |h_R(\tau,w,y,z)|\Big)$$

Indeed, (4.7), (4.13) and (4.14) imply that, if $t \in [t_i, t_{i+1})$ for some $i = 0,\dots,n(P) - 1$, then

$$|W_P^+(x,t) - W_{P,R}^{+,n}(x,t)| \le$$

$$\le \sup_{z\in Z} \sup_{y\in Y} |e^{(t_{i+1}-t)g(t,x,y,z)} W_P^+(x + (t_{i+1} - t)f(t,x,y,z), t_{i+1})$$

$$-e^{(t_{i+1}-t)g^n(t,x,y,z)} W_{P,R}^{+,n}(x + (t_{i+1} - t)f_R(t,x,y,z), t_{i+1})| +$$

$$+ (t_{i+1} - t) \sup_{z\in Z} \sup_{y\in Y} |h(t,x,y,z) - h^n(t,x,y,z)|$$

and thus

$$|W_P^+(x,t) - W_{P,R}^{+,n}(x,t)| \le$$

$$\le e^{(t_{i+1}-t)B_R} \sup_{z\in Z} \sup_{y\in Y} |W_P^+(x + (t_{i+1} - t)f(t,x,y,z), t_{i+1}) -$$

$$- W_{P,R}^{+,n}(x + (t_{i+1} - t)f(t,x,y,z), t_{i+1})|$$

$$+ e^{(t_{i+1}-t)B_R} W_R(t_{i+1} - t)\frac{1}{n} + (t_{i+1} - t)\frac{1}{n}$$

A simple inductive argument then implies (4.15) for "+" case. The "-" case follows exactly the same way.

Now we define $H^{+,n}, H^{-,n} : [0,T] \times R^N \times R \times R^N \to R$ by

$$H^{+,n}(\tau,w,r,p) = \inf_{z\in Z} \sup_{y\in Y} \{f_R(\tau,w,y,z)\cdot p + g^n(\tau,w,y,z)r + h^n(\tau,w,y,z)\}$$

and

$$H^{-,n}(\tau,w,r,p) = \sup_{y\in Y} \inf_{z\in Z} \{f_R(\tau,w,y,z)\cdot p + g^n(\tau,w,y,z)r + h^n(\tau,w,y,z)\}$$

and consider the problems

$$(4.16)^{\pm}\qquad \begin{cases} \dfrac{\partial u^{\pm,n}}{\partial \tau} + H^{\pm,n}(\tau,w,u^{\pm,n},Du^{\pm,n}) = 0 & \text{in } \mathbb{R}^N \times [0,T) \\ u^{\pm,n}(w,T) = u_0^n(w) & \text{in } \mathbb{R}^N \end{cases}$$

Since $H^{\pm,n}$ obviously satisfy (H1), (H2), (H4) and (H5) and $u_{0,R} \in BUC(\mathbb{R}^N)$, in view of Theorem 1.1 (appropriately modified so that it applies for the reverse time problem), we have that, for each n, $(4.16)^{\pm}$ has a unique viscosity solution $u^{\pm,n} \in BUC(\bar{Q}_T)$. Moreover, in view of Proposition 1.5 (a), (c), the definition of u_0^n and the fact that $H^{+,n}$ and $H^{-,n}$ satisfy (H2) and (H4) with the same constants, there are constants R_n and C_n such that

$$\|u^{\pm,n}\| \le R_n$$

and

$$\sup_{0\le\tau\le T} \|Du^{\pm,n}(\cdot,\tau)\| \le C_n$$

But then Proposition 1.4 implies that, for every (w,τ) in $\bar{Q}_T$, it is

$$(4.17)\qquad |u^{+,n}(w,\tau) - u^{-,n}(w,\tau)| \le Te^{TB_R} \sup_{\substack{\bar{\tau}\in[0,T] \\ \bar{w}\in\mathbb{R}^N \\ |r|\le R_n \\ |p|\le C_n}} |H^{+,n}(\bar{\tau},\bar{w},r,p) - H^{-,n}(\bar{\tau},\bar{w},r,p)|$$

Finally, by Theorem 3.1 and Remark 3.2 (modified so they apply to the reverse time problem), for each n, we have that

$$(4.18)\qquad \|W_{P,R}^{\pm,n} - u^{\pm,n}\| \to 0 \quad \text{as} \quad |P| \to 0$$

Now let $\varepsilon > 0$ be fixed but arbitrary and choose n large enough so that

$$\frac{2}{n} e^{TB_R}[T(W_R + 1) + 1] < \frac{\varepsilon}{2}$$

For such an n, let $\rho_0 > 0$ be so that, if $|P|,|Q| < \rho_0$, then

$$\|W_{P,R}^{\pm,n} - W_{Q,R}^{\pm,n}\| < \frac{\varepsilon}{2}$$

But then, in view of (4.15), we have that, for $|P|,|Q| < \rho_0$, it is

$$|W_P^{\pm}(x,\tau) - W_Q^{\pm}(x,t)| < \varepsilon$$

This implies that there exists $W^{\pm}(x,t)$ such that

$$\lim_{|P|\to 0} W_P^{\pm}(x,t) = W^{\pm}(x,t)$$

i.e. the upper and lower value exist.

For the existence of the value observe that, if the Isaacs condition holds, then, in view of (4.13) and the definition of $H^{\pm}$, it is

$$\sup_{\substack{\tau\in[0,T] \\ w\in R^N \\ |r|\le R_n \\ |p|\le C_n^n}} |H^{+,n}(\tau,w,r,p) - H^{-,n}(\tau,w,r,p)| \le \frac{2}{n}(1 + R_n)$$

Therefore by (4.17) we have

$$\|u^{+,n} - u^{-,n}\| \le \frac{2}{n}\left(Te^{B_R T}(1 + R_n)\right) \tag{4.19}$$

The last observation is that, since $\|u_0^n\| \le \|u_{0,R}\|$ and $|h_R(\tau,w,y,z) - h^n(\tau,w,y,z)| \le \frac{1}{n}$, then

$$R_n \le Te^{B_R T}(\|u_{0,R}\| + T \sup_{(\tau,w,y,z)} |h_R(\tau,w,y,z)|) < \infty$$

Now for $\varepsilon > 0$ fixed but arbitrary let n be large enough so that

$$\frac{2}{n}\left(Te^{B_R T}(1 + R_n)\right) + \frac{2}{n} e^{B_R T}[T(W_R + 1) + 1] < \frac{\varepsilon}{2}$$

Having chosen n as above, let $\rho_0 > 0$ be so that, if $|P| < \rho_0$, then

$$|W_P^+(x,\tau) - W^+(x,\tau)| + |W_P^-(x,\tau) - W^-(x,\tau)| + \|W_{P,R}^{+,n} - u^{+,n}\| + \|W_{P,R}^{-,n} - u^{-,n}\| < \frac{\varepsilon}{2}$$

We have

$$|W^+(x,\tau) - W^-(x,\tau)| < \varepsilon$$

and thus the result.

For the last part observe that in the case that f,g,h and u_0 satisfy (FL4) and (FL5), then so do f_R, g_R, h_R and $u_{0,R}$. Moreover, if we define $W_{P,R}^+$, $W_{P,R}^-$: $\bar{Q}_T \to R$ by (4.7) and (4.8) using f_R, g_R, h_R and $u_{0,R}$, then, for every $(x,t) \in \bar{Q}_T$ with $|x| \le \bar{R}$,

it is

$$W^{\pm}_{P,R}(x,t) = W^{\pm}_{P}(x,t) \ .$$

Finally, we can apply Theorem 3.1 (b) to the problems

$$\begin{cases} \dfrac{\partial u^{\pm}_R}{\partial t} + H^{\pm}_R(t,w,u^{\pm}_R,Du^{\pm}_R) = 0 \quad \text{in} \quad R^N \times [0,T) \\ u^{\pm}_R(w,T) = u_{0,R}(w) \quad \text{in} \quad R^N \end{cases}$$

and the functions $W^{\pm}_{P,R}$ to obtain that, for $|P|$ sufficiently small,

$$\| W^{\pm}_{P,R} - u^{\pm}_R \| \le K|P|^{1/2}$$

where $u^{\pm}_R$ is the viscosity solution of the above problem (obtained by Theorem 1.1) and K is a constant which depends on $\| u_{0,R} \|$, $\| Du_0 \|$ and $\bar{R}$. Then for $|x| < \bar{R}$ we have that

$$|W^{\pm}_P(x,t) - u^{\pm}_R(x,t)| \le K|P|^{1/2}$$

Since as $|P| \to 0$, $W^{\pm}_P(x,t) \to u^{\pm}_R(x,t)$, it is

$$W^{\pm}(x,t) = u^{\pm}_R(x,t)$$

and thus the result.

As a corollary of the above theorem and its proof we can obtain the following proposition

<u>Proposition 3.1</u>. Suppose that f satisfies (FL1) and that h,g and $u_0 \in BUC(\mathbb{R}^N)$. Then $W^{\pm}(x,t)$ exists for every $(x,t) \in \bar{Q}_T$. Moreover, it is a viscosity solution of the problem

$$\begin{cases} \dfrac{\partial W^{\pm}}{\partial t} + H^{\pm}(t,x,W^{\pm},DW^{\pm}) = 0 \quad \text{in} \quad R^N \times [0,T) \\ W^{\pm}(x,T) = u_0(x) \quad \text{in} \quad R^N \end{cases}$$

where $H^{\pm} : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is given by (4.5). Finally, if the Isaacs condition holds, then the value $W(x,t)$ of the differential game exists for every $(x,t) \in \bar{Q}_T$ and it is a viscosity solution of the Isaacs-Bellman equation

$$\begin{cases} \dfrac{\partial W}{\partial t} + H(t,x,W,DW) = 0 & \text{in } \mathbf{R}^N \times [0,T) \\ W(x,T) = u_0(x) & \text{in } \mathbf{R}^N \end{cases}$$

where H is the Hamiltonian function of the differential game.

<u>Proof</u>. The existence of $W^{\pm}$ for every $(x,t) \in \bar{Q}_T$ follows from Theorem 4.1. Moreover, it is easy to check using (4.7) and (4.8) that $W^{\pm} \in C(\bar{Q}_T)$. Finally, in the course of the proof of Theorem 4.1, we showed that if $|x| < \bar{R}$ for some $\bar{R} > 0$, then

$$W^{\pm}(x,t) = \lim_{n\to\infty} u_R^{\pm,n}(x,t)$$

where $u_R^{\pm,n}$ is the viscosity solution of an appropriately defined problem.

Now if for $\phi \in C^{\infty}(\mathbf{R}^N)$, $W^{+} - \phi$ attains a local maximum (which without any loss of generality can be assumed to be strict) at $(x_0,t_0) \in \mathbf{R}^N \times [0,T)$, we claim that

$$\frac{\partial \phi}{\partial t}(x_0,t_0) + H^{+}(t_0,x_0,W^{+}(x_0,t_0),D\phi(x_0,t_0)) \geq 0 \;^{(*)}$$

Indeed let $\bar{R}_0$ be such that

$$|x_0| < \bar{R}_0$$

and choose R_0 using (4.11). Then we know that for $|x| < \bar{R}_0$

$$W^{+}(x,t) = \lim_{n\to\infty} u_{R_0}^{+,n}(x,t)$$

with the limit uniform in (x,t). But there are (x_n,t_n) in $\mathbf{R}^N \times [0,T)$ with $|x_n| < \bar{R}_0$ such that $u_{R_0}^{+,n} - \phi$ attains a local maximum at (x_n,t_n) and, moreover, as $n \to \infty$

$$(x_n,t_n) \to (x_0,t_0)$$

Since it is

$$\frac{\partial \phi}{\partial t}(x_n,t_n) + H^{+,n}(t_n,x_n,u_{R_0}^{+,n}(x_n,t_n),D\phi(x_n,t_n)) \geq 0 \;^{(**)}$$

in view of the definition of $H^{+,n}$, as $n \to \infty$

$$\frac{\partial \phi}{\partial t}(x_0,t_0) + H^{+}(t_0,x_0,W^{+}(x_0,t_0),D\phi(x_0,t_0)) \geq 0$$

(*)(**)This inequality corresponds to (1.1), if one solves the reverse time problem.

Similar observations for the case that $W^{+} - \phi$ attains a strict local minimum at (x_0, t_0) imply the result for W^{+}. For W^{-} one repeats the above arguments.

(b) <u>The value in the sense of Friedman</u>

We begin with the following assumptions on f, h and u_0 (here for simplicity we take $g \equiv 0$)

$$
\text{(FR1)} \quad \begin{cases} f \in C([0,T] \times \mathbb{R}^N \times Y \times Z). \text{ Moreover, there exists} \\ \text{a constant } K_f \text{ such that, for every } t \in [0,T], \\ y \in Y, \ z \in Z \text{ and } x, \bar{x} \in \mathbb{R}^N, \text{ it is} \\ \qquad |f(t,x,y,z) - f(t,\bar{x},y,z)| \le K_f |x - \bar{x}| \end{cases}
$$

(FR2) $h \in C([0,T] \times \mathbb{R}^N \times Y \times Z)$

(FR3) $u_0 \in C(\mathbb{R}^N)$

(FR1) implies that, for $y : [0,T] \to Y$ and $z : [0,T] \to Z$ measurable functions and $(x,t) \in \bar{Q}_T$, the system

$$
(4.20) \qquad \begin{cases} \dfrac{d\xi}{d\tau} = f(\tau, \xi(\tau), y(\tau), z(\tau)) & t < \tau \le T_1 \le T \\ \xi(t) = x \end{cases}
$$

has a unique solution, which we denote by $\xi(\tau; x, t, T_1, y, z)$

For a partition $P = \{0 = t_0 < t_1 < \ldots < t_{n(P)} = T\}$ of $[0,T]$, let $V_P^{+}, V_P^{-} : \bar{Q}_T \to \mathbb{R}$ be defined by

$$
(4.21)\quad \begin{cases} V_P^+(x,T) = u_0(x) \\[2ex] V_P^+(x,t) = \displaystyle\inf_{z\in Z(t,t_{i+1})}\ \sup_{y\in Y(t,t_{i+1})} \Big\{V_P^+(\xi(t_{i+1};x,t,t_{i+1},y,z),t_{i+1}) \\[2ex] \qquad + \displaystyle\int_t^{t_{i+1}} h(s;\xi(s,x,t,t_{i+1},y,z),y(s),z(s))ds\Big\} \\[2ex] \text{if } \ t \in [t_i,t_{i+1}) \ \text{ for } \ i = 0,\dots,n(P) - 1 \end{cases}
$$

and

$$
(4.22)\quad \begin{cases} V_P^-(x,T) = u_0(x) \\[2ex] V_P^-(x,t) = \displaystyle\sup_{y\in Y(t,t_{i+1})}\ \inf_{z\in Z(t,t_{i+1})} \Big\{V_P^-(\xi(t_{i+1};x,t,t_{i+1},y,z),t_{i+1}) + \\[2ex] \qquad + \displaystyle\int_t^{t_{i+1}} h(s,\xi(s;x,t,t_{i+1},y,z),y(s),z(s))ds\Big\} \\[2ex] \text{if } \ t \in [t_i,t_{i+1}) \ \text{ for } \ i = 0,\dots,n(P) - 1 \end{cases}
$$

where for $0 \le \tau < \hat{\tau} \le T$, $Y(\tau,\hat{\tau})$ denotes the set of *measurable functions* $y : [\tau,\hat{\tau}] \to Y$ and $Z(\tau,\hat{\tau})$ denotes the set of measurable functions $z : [\tau,\hat{\tau}] \to Z$, (It is assumed that y, z are defined only almost everywhere) and for $y \in Y(\tau,\hat{\tau})$ and $z \in Z(\tau,\hat{\tau})$, $\xi(\cdot;x,\tau,\hat{\tau},y,z)$ denotes the unique solution of

$$
\begin{cases} \dfrac{d\xi}{ds} = f(x,\xi(s),y(s),z(s)) & \tau < s \le \hat{\tau} \\[2ex] \xi(\tau) = x \end{cases}
$$

It follows from (4.21), (4.22) and Lemma 1.4 of [16] that for every $(x,t) \in \bar{\Omega}_T$, $V_P^+(x,t)$, $(V_P^-(x,t))$ is the upper (lower) P-value of the differential game given by (4.1) and (4.4) with initial condition (x,t). (This concept of upper (lower) P-value of a differential game was introduced by Friedman in [15]. Since it is rather lengthy, we do not explain it here). The question again is whether $\lim_{|P|\to 0} V_P^+(x,t)$ and $\lim_{|P|\to 0} V_P^-(x,t)$

exist and if yes whether the limits are equal. Before we answer it, we need the following definition.

Definition 4.2. (A. Friedman [15], [16]) For a partition P of $[0,T]$ and $(x,t) \in \bar{Q}_T$ let $V_P^+(x,t)$ and $V_P^-(x,t)$ be defined by (4.21) and (4.22) respectively. If $\lim_{|P|\to 0} V_P^+(x,t) = V^+(x,t)$ $\left(\lim_{|P|\to 0} V_P^-(x,t) = V^-(x,t)\right)$ exists, then $V^+(x,t)$ $(V^-(x,t))$ is the upper (lower) value of the differential game with dynamics (4.1), payoff (4.4) and initial condition (x,t). If $V^+(x,t) = V^-(x,t)$, then $V(x,t) = V^+(x,t) = V^-(x,t)$ is the value of this differential game.

Now we state and prove the theorem, which establishes the existence of the upper and lower value and, under the Isaacs condition, the existence of the value (in the sense of Friedman). We have

Theorem 4.2. If (FR1), (FR2) and (FR3) are satisfied, then, for every $(x,t) \in \bar{Q}_T$, the upper (lower) value $V^+(x,t)$ $(V^-(x,t))$ of the differential game with dynamics (4.1), payoff functional (4.4) $(g \equiv 0)$ and initial condition (x,t) exists. Moreover, if the Isaacs condition holds, then the value $V(x,t)$ exists.

Remark 4.2. Theorem 4.2 was proved by Friedman ([16]). His method is related to theory of stochastic differential games. Here we give a direct proof using the results of Section 2.

Proof of Theorem 4.2. Using the arguments at the beginning of the proof of Theorem 4.1 we can easily reduce to the case where

(FR4) $\left\{\begin{array}{l} \text{If } \psi \text{ is any of the functions } f \text{ and } h, \text{ then } \psi \text{ is uniformly} \\ \text{continuous in } (t,x,y,z) \text{ and moreover, there exist constants } K_\psi \text{ and} \\ B_\psi, \text{ so that, for every } t \in [0,T],\ x,\bar{x} \in R^N,\ y \in Y \text{ and } z \in Z, \text{ it is} \\ \qquad |\psi(t,x,y,z) - \psi(t,\bar{x},y,z)| \le K_\psi |x - \bar{x}| \\ \text{and} \\ \qquad |\psi(t,x,y,z)| \le B_\psi \end{array}\right.$

and

(FR5) $u_0 \in BUC(\mathbb{R}^N)$.

Here we only prove the existence of the upper value, since for the lower value one uses the same arguments. To this end, observe that if H^+ is given by (4.5) (with $g \equiv 0$), then, in view of our assumptions, H^+ satisfies (H1), (H2), (H4) and (H5). Thus the (reverse time) problem

$$(4.23)\qquad \begin{cases} \dfrac{\partial V^+}{\partial t} + H^+(t,x,DV^+) = 0 & \text{in } \mathbb{R}^N \times [0,T) \\ V^+(x,T) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

has a unique viscosity solution V^+ in $\bar{Q}_T$. We claim that as $|P| \to 0$

$$\|V_P^+ - V^+\| \to 0$$

i.e. that the upper value exists and it is the unique viscosity solution of (4.24). To prove the claim we are going to use Theorem 2.1 appropriately modified, so it applies to the reverse time problem. In particular, for $(t,\rho) \in K = \{(t,\rho) \in [0,T] \times [0,T] : 0 \le \rho \le T - t\}$, and $u \in BUC(\mathbb{R}^N)$, let $F(t,\rho,u) : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$(4.24)\qquad \begin{cases} F(t,\rho,u)(x) = \inf\limits_{z \in Z(t,t+\rho)} \sup\limits_{y \in Y(t,t+\rho)} \{u(\xi(t+\rho;x,t,t+\rho,y,z)) + \\ \displaystyle\int_t^{t+\rho} h(s,\xi(s;x,t,t+\rho,y,z),y(s),z(s))ds\} \end{cases}$$

The fact that $F(t,\rho,u) \in BUC(\mathbb{R}^N)$ and, moreover, that (F1), (F2) (for $u \in C_b^{0,1}(\mathbb{R}^N)$), (F3), (F4) (for $u \in C_b^{0,1}(\mathbb{R}^N)$), (F9), (F10) and (F11) are satisfied is an easy consequence of (4.24), the assumptions on f,h and u_0 and the properties of the solutions of (4.21). Here we only check (F12) (in particular its modification for the inverse time problem), since it is somehow more involved. We claim that for $\phi \in C_b^2(\mathbb{R}^N)$ it is

$$(4.25)\qquad \left\|\frac{F(t,\rho,\phi) - \phi}{\rho} - H^+(t,\cdot,D\phi)\right\| \to 0 \quad \text{as } \rho \to 0$$

To prove the above let us define $A(t,x,\rho)$ and $A(t,x)$ by

$$(4.26)\qquad \begin{cases} A(t,x,\rho) = \dfrac{F(t,\rho,\phi)(x) - \phi(x)}{\rho} \\ A(t,x) = \inf\limits_{z \in Z} \sup\limits_{y \in Y} \{f(t,x,y,z)\cdot D\phi(x) + h(t,x,y,z)\} \end{cases}$$

Then (4.25) is equivalent to

$$(4.27)\qquad \begin{cases} A(t,x,\rho) \to A(t,x) \quad \text{as} \quad \rho \to 0 \quad \text{with} \\ \text{the limit uniform in} \quad (t,x) \ . \end{cases}$$

Suppose that (4.27) is not true. Then there are $\varepsilon_0 > 0$ and x_ρ, t_ρ such that as $\rho \to 0$

$$(4.28)\qquad |A(t_\rho,x_\rho,\rho) - A(t_\rho,x_\rho)| > \varepsilon_0$$

We are going to show that (4.28) leads to a contradiction. To this end, let $\Lambda : [0,T] \times \mathbb{R}^N \times Y \times Z \to \mathbb{R}$ be defined by

$$(4.29)\qquad \Lambda(t,x,y,z) = f(t,x,y,z)\cdot D\phi(x) + h(t,x,y,z) \ .$$

In view of the properties of f, h and ϕ, Λ is uniformly continuous in (t,x,y,z). This implies that there is a $\delta_1 > 0$ such that if, for $(\tau,x,y,z),(\bar{\tau},\bar{x},\bar{y},\bar{z}) \in [0,T] \times \mathbb{R}^N \times Y \times Z$, it is $\max\{|\tau - \bar{\tau}|,|x - \bar{x}|,|y - \bar{y}|,|z - \bar{z}|\} < \delta_1$, then

$$(4.30)\qquad |\Lambda(\bar{\tau},\bar{x},\bar{y},\bar{z}) - \Lambda(\tau,x,y,z)| < \varepsilon_0/4$$

Next let $y \in Y(\tau,\bar{\tau})$ and $z \in Z(\tau,\bar{\tau})$. Then for $s \in [\tau,\bar{\tau}]$ it is immediate that

$$(4.31)\qquad |\xi(s;x,\tau,\bar{\tau},y,z) - x| < (s - \tau)B_f$$

So if $\rho_0 > 0$ is chosen to be

$$\rho_0 = \min\{\delta_1,\ \delta_1/B_f\}$$

for $t < s < t + \rho < t + \rho_0$, we have

$$(4.32)\qquad |\Lambda(s,\xi(s;x,t,t+\rho,y,z),\bar{y},\bar{z}) - \Lambda(t,x,\bar{y},\bar{z})| < \varepsilon_0/4$$

for every $(\bar{y},\bar{z}) \in Y \times Z$ and $(y(\cdot),z(\cdot)) \in Y(t,t+\rho_0) \times Z(t,t+\rho_0)$.

We have to examine the following two cases:

<u>Case 1</u>. Along some subsequence $\rho_k \downarrow 0$ it is

$$A(t_{\rho_k},x_{\rho_k},\rho_k) - \varepsilon_0 > A(t_{\rho_k},x_{\rho_k})$$

For each ρ (here for simplicity we denote the subsequence ρ_k again as ρ), in view of

(4.26) and (4.29), we have

$$\Lambda(t_\rho,x_\rho,\rho) - \varepsilon_0 > \inf_{z\in Z} \sup_{y\in Y} \Lambda(t_\rho,x_\rho,y,z)$$

Since $\Lambda(t_\rho,x_\rho,y,z)$ is uniformly continuous in (y,z) and Y,Z are compact sets, we can find $z_\rho \in Z$ such that

$$\Lambda(t_\rho,x_\rho,\rho) - \varepsilon_0 > \Lambda(t_\rho,x_\rho,\bar{y},z_\rho) \quad \text{for every } \bar{y} \in Y$$

If $\rho < \rho_0$, then (4.32) implies that, for every $t_\rho < s < t_\rho + \rho$ and $(y,z) \in Y(t_\rho,t_\rho + \rho) \times Z(t_\rho,t_\rho + \rho)$, it is

$$\Lambda(t_\rho,x_\rho,\rho) - \frac{3}{4}\varepsilon_0 > \Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z),\bar{y},z_\rho) \quad \text{for every } \bar{y} \in Y \tag{4.33}$$

Therefore, for every $s \in [t_\rho,t_\rho + \rho]$ and $y(\cdot) \in Y(t_\rho,t_\rho + \rho)$, if $z_\rho(\cdot) \in Z(t_\rho,t_\rho + \rho)$ is defined so that $z_\rho(s) = z_\rho$, (4.33) implies

$$\Lambda(t_\rho,x_\rho,\rho) - \frac{3}{4}\varepsilon_0 > \Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z_\rho),y(s),z_\rho(s))$$

Integrating both sides of the above inequality over $(t_\rho,t_\rho + \rho)$ we obtain

$$\begin{aligned}
&\Lambda(t_\rho,x_\rho,\rho) - \frac{3}{4}\varepsilon_0 > \\
&> \frac{1}{\rho}\int_{t_\rho}^{t_\rho+\rho} f(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z_\rho)y(s),z_\rho(s))\cdot D\phi(\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z_\rho))ds \\
&+ \frac{1}{\rho}\int_{t_\rho}^{t_\rho+\rho} h(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z_\rho),y(s),z_\rho(s))ds
\end{aligned}$$

Therefore, for every $y \in Y(t_\rho,t_\rho + \rho)$, it is

$$\begin{aligned}
\Lambda(t_\rho,x_\rho,\rho) - \frac{3}{4}\varepsilon_0 &> \frac{1}{\rho}(\phi(\xi(t_\rho + \rho;x_\rho,t_\rho,t_\rho + \rho,y,z_\rho)) - \phi(x_\rho)) + \\
&+ \frac{1}{\rho}\int_{t_\rho}^{t_\rho+\rho} h(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z_\rho),y(s),z_\rho(s))ds
\end{aligned}$$

So

$$\Lambda(t_\rho,x_\rho,\rho) - \frac{3}{4}\varepsilon_0 > \frac{1}{\rho} \sup_{y\in Y(t_\rho,t_\rho+\rho)} \Big[\phi(\xi(t_\rho + \rho; x_\rho,t_\rho,t_\rho + \rho,y,z_\rho)) +$$

$$+ \int_{t_\rho}^{t_\rho+\rho} h(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z_\rho),y(s),z_\rho(s))ds] - \phi(x_\rho)\Big]$$

and finally

$$\Lambda(t_\rho,x_\rho,\rho) - \frac{3}{4}\varepsilon_0 > \frac{1}{\rho}\Big\{ \inf_{z\in Z(t_\rho,t_\rho+\rho)} \sup_{y\in Y(t_\rho,t_\rho+\rho)} \Big[\phi(\xi(t_\rho + \rho; x_\rho,t_\rho,t_\rho + \rho,y,z)) +$$

$$+ \int_{t_\rho}^{t_\rho+\rho} h(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y,z),y(s),z(s))ds - \phi(x_\rho)\Big]\Big\}$$

$$= \frac{1}{\rho}(F(t_\rho,x_\rho,\phi)(x_\rho) - \phi(x_\rho)) = \Lambda(t_\rho,x_\rho,\rho)$$

which contradicts the fact that $\varepsilon_0 > 0$.

<u>Case 2</u>. As $\rho \downarrow 0$, $\Lambda(t_\rho,x_\rho) > \varepsilon_0 + \Lambda(t_\rho,x_\rho,\rho)$.

For $\rho < \rho_0$ and since

$$\inf_{z\in Z} \sup_{y\in Y} \Lambda(t_\rho,x_\rho,y,z) > \varepsilon_0 + \Lambda(t_\rho,x_\rho,\rho)$$

for every $z \in Z$ we can find $y_\rho = y_\rho(z) \in Y$ so that

$$\Lambda(t_\rho,x_\rho,y_\rho(z),z) > \varepsilon_0 + \Lambda(t_\rho,x_\rho,\rho)$$

Then, in view of (4.32), for $t_\rho < s < t_\rho + \rho$ and $(\bar{y},\bar{z}) \in Y(t_\rho,t_\rho + \rho) \times Z(t_\rho,t_\rho + \rho)$ we have

(4.34) $\quad \Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,\bar{y},\bar{z}),y_\rho(z),z) > \frac{3}{4}\varepsilon_0 - \Lambda(t_\rho,x_\rho,\rho)$ for every $z \in Z$

For every $z(\cdot) \in Z(t_\rho,t_\rho + \rho)$ we can find a sequence $\{z_k\}$ of step functions defined on $[t_\rho,t_\rho + \rho]$, such that

(4.35) $$\begin{cases} \sup_{s \in I} |z(s) - z_k(s)| < \delta_1 & \text{for some } k \\ \text{meas } I < \frac{1}{16M} \varepsilon_0 \rho \end{cases}$$

where I is a subset of $[t_\rho, t_\rho + \rho]$ and $M > 0$ is such that

$$|\Lambda(\tau,w,y,z)| < M \quad \text{for every} \quad (\tau,w,y,z) \in [0,T] \times \mathbb{R}^N \times Y \times Z$$

Since

$$z_k(s) = z_{ki}$$

for $s \in B_{ki}$ $(i = 1,\dots,i_0 = i_0(k), \ \bigcup_i B_{ki} = [t_\rho, t_\rho + \rho])$, then, if $y_{ki} = y_{ki}(z_{ki})$, (4.34) implies

(4.35) $$\Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,\bar{y},z),y_{ki},z_{ki}) > \frac{3}{4}\varepsilon_0 - A(t_\rho,x_\rho,\rho)$$

Now define $y_k \in Y(t_\rho,t_\rho + \rho)$ by

$$y_k(s) = y_{ki} \quad \text{if} \quad s \in B_{ki}$$

Then (4.35) implies

(4.36) $$\Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y_k,z),y_k(s),z_k(s)) > \frac{3}{4}\varepsilon_0 + A(t_\rho,x_\rho,\rho)$$

Moreover

$$\Big|\int_{t_\rho}^{t_\rho+\rho} \Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y_k,z),y_k(s),z_k(s))$$

$$- \Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y_k,z),y_k(s),z(s))ds\Big|$$

$$< \frac{(\rho - \text{meas } I)}{4}\varepsilon_0 + 2(\text{meas } I)M < \frac{3\rho}{8}\varepsilon_0$$

This inequality together with (4.36) gives

$$\frac{1}{\rho}\int_{t_\rho}^{t_\rho+\rho} \Lambda(s,\xi(s;x_\rho,t_\rho,t_\rho + \rho,y_k,z),y_k(s),z(s))ds > \frac{3}{8}\varepsilon_0 + A(t_\rho,x_\rho,\rho)$$

Thus we proved that, for any control $z(\cdot) \in Z(t_\rho,t_\rho + \rho)$, there is a control $y(\cdot) \in Y(t_\rho,t_\rho + \rho)$ such that

$$\frac{1}{\rho}\Big[\phi(\xi(t_\rho + \rho; x_\rho, t_\rho, t_\rho + \rho, y, z)) +$$

$$+ \int_{t_\rho}^{t_\rho+\rho} h(s, \xi(s; x_\rho, t_\rho, t_\rho + \rho, y, z), y(s), z(s))ds - \phi(x_\rho)\Big] > \frac{3}{8}\varepsilon_0 + A(t_\rho, x_\rho, \rho)$$

therefore

$$A(t_\rho, x_\rho, \rho) > \frac{3}{8}\varepsilon_0 + A(t_\rho, x_\rho, \rho)$$

which contradicts the fact that $\varepsilon_0 > 0$.

Having shown that the upper and lower value exist, the fact that the value exists, if Isaacs condition holds, is an immediate consequence of the uniqueness of the viscosity solution.

Remark 4.3. Under assumptions (FR4) and (FR5) one can prove that the upper and lower value exist by using the fact that V_p^+ and V_p^- are monotone with respect to $|P|$ ([15]). Then, by modifying some of the arguments used by Friedman in [15] and the equation of dynamic programming, one can show directly that V^+ and V^- are viscosity solutions of the appropriate problems. This was first observed but not published by the author. (Later it appeared in [2]). Nevertheless the arguments we gave here in order to verify (F12) are again related to the equation of dynamic programming, which is hidden behind the recursive relation that defines $V_p^\pm$.

Remark 4.4. The fact that Friedman's and Fleming's definitions lead to the same value follows from the proofs of Theorems 4.1 and 4.2. Indeed in either case we showed that the upper (lower) value, under the most general assumptions, is obtained as the limit of the unique viscosity solutions of the same problems.

We conclude this section with some remarks about optimal control theory and a related theorem. An optimal control problem of the kind we consider here consists of

(1) an ordinary differential equation

$$\begin{cases} \dfrac{d\xi}{d\tau} = \xi(\tau,\xi(\tau),v(\tau)), & 0 \le t \le \tau \le T \\ \xi(t) = x \end{cases}$$

where Y is a compact subset of $\mathbb{R}^p$ for some p, $f : [0,T] \times \mathbb{R}^N \times Y \to \mathbb{R}^N$ satisfies (FR1) and $v : [0,T] \to \mathbb{R}$ is a measurable Y-valued function defined for almost every τ in $[0,T]$.

(2) a payoff functional $P(v)$ given by

$$P(v) = u_0(\xi(T;x,t,T,v)) + \int_t^T h(s,\xi(s;x,t,T,v),v(s))ds$$

where $u_0 \in C(\mathbb{R}^N)$, $h \in C([0,T] \times \mathbb{R}^N \times Y)$ and for $0 \le \tau \le \bar{\tau} \le T$, $\xi(\cdot;x,\tau,\bar{\tau},v)$ is the unique solution of

$$\begin{cases} \dfrac{d\xi}{ds} = f(s,\xi(s),v(s)) & \tau < s \le \hat{\tau} \\ \xi(\tau) = x \end{cases}$$

The aim is to minimize the value of the payoff functional over all possible control functions. The value of the problem is defined by:

$$v(x,t) = \inf_{v(\cdot)} [u_0(\xi(T;x,t,T,v)) + \int_t^T h(s,\xi(s;x,t,T,v),v(s))ds] \tag{4.37}$$

P. L. Lions ([18], also see L. C. Evans and I. Capuzzo Dolcetta [3]) observed that v is a viscosity solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} + \inf_{y \in Y} \{f(t,x,y)\cdot Dv + h(t,x,y)\} = 0 & \text{in } \mathbb{R}^N \times [0,T) \\ v(x,T) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{4.38}$$

Here we state a result which proves the convergence of a certain approximation scheme to the value of the optimal control problem. In particular, if, for a partition $P = \{0 = t_0 < t_1 < \ldots < t_{n(P)} = T\}$, *we define* $u_P : \bar{Q}_T \to \mathbb{R}$ *by*

$$\begin{cases} u_P(x,T) = u_0(x) \\ u_P(x,t) = \inf\limits_{y\in Y} \{u_P(x + (t_{i+1} - t)f(t,x,y),t_{i+1}) + (t_{i+1} - t)h(t,x,y)\} \\ \text{if} \quad t \in [t_i,t_{i+1}) \quad \text{for some} \quad i = 0,\dots,n(P) - 1 \end{cases}$$

then it is

<u>Theorem 4.3</u>. If f,h satisfy (FR4) and $u_0 \in BUC(\mathbb{R}^N)$, then, as $|P| \to 0$

$$\|u_P - v\| \to 0$$

where v is the value of the under consideration optimal control problem. If, moreover, f,h also satisfy (FL4) and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then, for $|P|$ sufficiently small,

$$\|u_P - v\| \le K|P|^{1/2}$$

where K is a constant which depends only on $\|u_0\|$, $\|Du_0\|$.

<u>Proof</u>. The proof is a direct consequence of Remark 3.2 and the fact that v is the *viscosity solution of* (4.38)

SECTION 5.

The first part of this section proves the convergence of explicit finite difference schemes to the viscosity solution of (0.1) and gives explicit error estimates. As mentioned in the introduction such a result was first proved by M. G. Crandall and P. L. Lions ([6]) for the problem (0.3). The theorem stated below is a generalization of their result.

We now describe the class of difference schemes to be considered here. For notational simplicity only, we will assume $N = 2$. The definitions and results for general N will be clear from this special case and we will not state them. A generic point in $\mathbb{R}^2$ will be denoted by (x,y) and we will write $Du = (u_x, u_y)$. Let α, β be some given positive numbers. For $\rho > 0$, $u : \mathbb{R}^2 \to \mathbb{R}$ and $(j,k) \in \mathbb{Z} \times \mathbb{Z}^{(*)}$, we define $u^{\rho}_{j,k} : \mathbb{R}^2 \to \mathbb{R}$, $\Delta^x_+ u^{\rho}_{j,k} : \mathbb{R}^2 \to \mathbb{R}$ and $\Delta^y_+ u^{\rho}_{j,k} : \mathbb{R}^2 \to \mathbb{R}$ by

$$(5.1)\qquad \begin{cases} u^{\rho}_{j,k}(x,y) = u(x + j\alpha\rho, y + k\beta\rho) \\ \Delta^x_+ u^{\rho}_{j,k}(x,y) = u^{\rho}_{j+1,k}(x,y) - u^{\rho}_{j,k}(x,y) \\ \Delta^y_+ u^{\rho}_{j,k}(x,y) = u^{\rho}_{j,k+1}(x,y) - u^{\rho}_{j,k}(x,y) \end{cases}$$

Moreover, for p,q,r,s fixed nonnegative integers and $u : \mathbb{R}^2 \to \mathbb{R}$ let $(\Delta^x_+ u)^{\rho} : \mathbb{R}^2 \to \mathbb{R}^{(p+q+1)(r+s+2)}$ and $(\Delta^y_+ u)^{\rho} : \mathbb{R}^2 \to \mathbb{R}^{(p+q+2)(r+s+1)}$ be defined by

$$(5.2)\qquad \begin{cases} (\Delta^x_+ u)^{\rho}(x,y) = (\Delta^x_+ u^{\rho}_{-p,-r}(x,y), \ldots, \Delta^x_+ u^{\rho}_{q,s+1}(x,y)) \\ \text{and} \\ (\Delta^y_+ u)^{\rho}(x,y) = (\Delta^y_+ u^{\rho}_{-p,-r}(x,y), \ldots, \Delta^y_+ u^{\rho}_{q+1,s}(x,y)) \end{cases}$$

If $u \in C^{0,1}(\mathbb{R}^2)$, it is easy to see that, for every $(x,y) \in \mathbb{R}^2$, it is

$$(5.3)\qquad \frac{|\Delta^x_+ u^{\rho}_{j,k}(x,y)|}{\rho\alpha}, \frac{|\Delta^y_+ u^{\rho}_{j,k}(x,y)|}{\rho\beta} < \|Du\|$$

(*) $\mathbb{Z}$ is the set of integers

and

$$(5.4)\qquad \frac{|(\Delta_+^x u)^\rho(x,y)|}{\rho\alpha},\ \frac{|(\Delta_+^y u)^\rho(x,y)|}{\rho\beta} \le A\|Du\|$$

where $A = \sqrt{2(p+q+2)(r+s+2)}$ and $|\ |$ denotes the usual metric in any $\mathbb{R}^m$.

Finally, for $u,v : \mathbb{R}^2 \to \mathbb{R}$ and $0 \le \rho \le t \le T$, let $F(t,\rho,u,v) : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$(5.5)\quad \begin{cases} F(t,\rho,u,v)(x,y) = v(x,y) - \rho g(t,x,y,u(x,y), \dfrac{(\Delta_+^x v)^\rho}{\rho\alpha}(x,y), \dfrac{(\Delta_+^y v)^\rho}{\rho\beta}(x,y)) & \text{if } \rho > 0 \\ \text{and} \\ F(t,0,u,v)(x,y) = v(x,y) \end{cases}$$

where $g : [0,T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)} \to \mathbb{R}$ satisfies

$$(G1)\quad \begin{cases} g \text{ is uniformly continuous on } [0,T] \times \mathbb{R}^2 \times [-R,R] \times \\ \times B_{(p+q+1)(r+s+2)}(0,R) \times B_{(p+q+2)(r+s+1)}(0,R) \quad \text{for every } R > 0 \end{cases}$$

$$(G2)\quad \begin{cases} \text{There exists a constant } C > 0 \text{ such that} \\ \qquad \sup_{(x,t)\in\bar{Q}_T} |g(t,x,0,0,\dots,0)| \le C \end{cases}$$

$$(G3)\quad \begin{cases} \text{For every } R > 0 \text{ there exists a constant } \bar{L}_R > 0 \text{ such that} \\ \qquad |g(t,x,y,r,w,z) - g(t,x,y,\bar{r},w,z)| \le \bar{L}_R|r - \bar{r}| \\ \text{for every } t \in [0,T],\ (x,y) \in \mathbb{R}^2,\ r,\bar{r} \in [-R,R] \\ \text{and } (w,z) \in \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)} \end{cases}$$

$$(G4)\quad \begin{cases} \text{For every } R > 0 \text{ there is a constant } C_R \text{ such that} \\ \qquad |g(t,x,y,r,w,z) - g(t,\bar{x},\bar{y},r,w,z)| \\ \qquad\qquad \le C_R(1 + |(w,z)|)(|t - \bar{t}| + |(x,y) - (\bar{x},\bar{y})|) \\ \text{for } t,\bar{t} \in [0,T],\ (x,y),(\bar{x},\bar{y}) \in \mathbb{R}^2,\ |r| \le R \\ \text{and } (w,z) \in \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)} \end{cases}$$

$$
\text{(G5)} \begin{cases} \text{For every } R > 0 \text{ there is a constant } M_R > 0 \text{ such that} \\ \qquad |g(t,x,y,r,w,z) - g(t,x,y,r,\bar{w},\bar{z})| \le M_R |(w,z) - (\bar{w},\bar{z})| \\ \text{for } t \in [0,T],\ (x,y) \in \mathbb{R}^2,\ |r| \le R \text{ and} \\ (w,z),(\bar{w},\bar{z}) \in \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)} \text{ with } |(w,z)|,|(\bar{w},\bar{z})| \le R \end{cases}
$$

The explicit finite difference schemes of interest here are generated by (5.5). We say that (5.5) is <u>consistent</u> with the equation $u_t + H(t,x,y,u,u_x,u_y) = 0$ occurring in (0.1), if

$$
(5.6) \qquad \begin{cases} g(t,x,y,r,a,\ldots,a,b,\ldots,b) = H(t,x,y,r,a,b) \text{ for} \\ \quad t \in [0,T],\ (x,y) \in \mathbb{R}^2,\ r \in \mathbb{R},\ a,b \in \mathbb{R} \end{cases}
$$

Moreover, we call (5.5) <u>monotone</u> on $[-R,R]$, if

$$
(5.7) \begin{cases} \text{For every } u : \mathbb{R}^2 \to \mathbb{R}, \text{ if } v(x,y) \le w(x,y) \text{ for every} \\ (x,y) \in \mathbb{R}^2, \text{ then, for any } (a,b) \in \mathbb{R}^2, \text{ such that} \\ \dfrac{|\Delta_+^{x}v^{\rho}_{j,k}(a,b)|}{\rho\alpha}, \dfrac{|\Delta_+^{y}v^{\rho}_{j',k'}(a,b)|}{\rho\beta}, \dfrac{|\Delta_+^{x}w^{\rho}_{j,k}(a,b)|}{\rho\alpha}, \dfrac{|\Delta_+^{y}w^{\rho}_{j',k'}(a,b)|}{\rho\beta} \le R \\ \text{for } -p \le j' \le q+1,\ -r \le k' \le s,\ -p \le j \le q,\ -r \le k \le s+1, \\ \text{it is } F(t,\rho,u,v)(a,b) \le F(t,\rho,u,w)(a,b). \end{cases}
$$

The main result is

<u>Theorem 5.1</u>. Let $H : [0,T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $u_0 \in C_b^{0,1}(\mathbb{R}^2)$. Let $g : [0,T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)} \to \mathbb{R}$ satisfy (G1), (G2), (G3), (G4) and (G5) and suppose that (5.5) is consistent with (0.1) and monotone on $[-(e^{T(2\bar{C}e^{\bar{L}T}+\bar{L})}(\|Du_0\| + \bar{C}T) + 1), e^{T(2\bar{C}e^{\bar{L}T}+\bar{L})}(\|Du_0\| + \bar{C}T) + 1]$, where, if $R = e^{\bar{L}T}(\|u_0\| + TC)$, then $\bar{C} = C_R$. For a partition P of $[0,T]$, define $u_P : \bar{Q}_T \to \mathbb{R}$ by (2.1) and (5.5). Let u be the viscosity solution of (0.1). Then there is a constant K, which depends only on $\|u_0\|$, $\|Du_0\|$, g and T, such that, for sufficiently small $|P|$,

$$
(5.8) \qquad \|u_P - u\| \le K|P|^{1/2}
$$

Proof: It suffices to check the assumptions of Theorem 2.1 (a). It is obvious that, for every (t,ρ) and $u,v \in C_b^{0,1}(\mathbb{R}^2)$, $F(t,\rho,u,v) \in C_b^{0,1}(\mathbb{R}^2)$. Moreover, (F1) follows from (5.5). Next, for $u \in C_b^{0,1}(\mathbb{R}^2)$ observe that

$$|F(t,\rho,u,u)(x,y) - u(x,y)| = \rho\,|g(t,x,y,u(x,y), \frac{(\Delta_+^x u)^\rho}{\rho\alpha}(x,y), \frac{(\Delta_+^y u)^\rho}{\rho\beta}(x,y))|$$

and therefore

$$\|F(t,\rho,u,u) - u\| \le \rho C_4(\|u\|,\|Du\|)$$

where $C_4(\|u\|,\|Du\|) = \sup\limits_{\substack{t\in[0,T]\\ \xi\in R^2\\ |r|\le\|u\|\\ |w|\le A\|Du\|\\ |z|\le A\|Du\|}} |g(t,\xi,r,w,z)|$

with A given by (5.4). The fact that $(t,\rho) \to F(t,\rho,u,u)$ is continuous in the $\|\ \|$-norm for $u \in C_b^{0,1}(\mathbb{R}^2)$ follows from the above inequality and (G2), (G3), (G4), (G5). Finally, (F3) is satisfied in view of (5.5).

Now we want to verify (F5). To this end, let

$$r = \sqrt{2}\,(\max\{p,r,q+1,s+1\} + 1)\max\{\alpha,\beta\}$$

and assume that $v(x,y) \le w(x,y)$ for every $(x,y) \in \mathbb{R}^2$. If for some $(\bar{x},\bar{y}) \in \mathbb{R}^2$ it is

$$|v(\bar{x}+a,\bar{y}+b) - v(\bar{x}+\bar{a},\bar{y}+\bar{b})|, |w(\bar{x}+a,\bar{y}+b) -$$

(5.9)

$$- w(\bar{x}+\bar{a},\bar{y}+\bar{b})| \le \bar{\bar{L}}|(a,b) - (\bar{a},\bar{b})|$$

for $(a,b),(\bar{a},\bar{b}) \in B_2(0,\rho r)$ and $\bar{\bar{L}} = e^{T(\bar{L}+2\bar{C}e^{\bar{L}T})}(\|Du_0\| + \bar{C}T) + 1$, we claim that, for any $u \in C_b^{0,1}(\mathbb{R}^N)$,

(5.10) $$F(t,\rho,u,v)(\bar{x},\bar{y}) \le F(t,\rho,u,\bar{v})(\bar{x},\bar{y}) .$$

This together with the fact, that, in view of (G2), (G3), (G4), (5.6) and Proposition 1.5 (c), $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^2)$ for every τ with

$$\sup_{0\le\tau\le T} \|Du(\cdot,\tau)\| \le e^{T(2\bar{C}e^{\bar{L}T}+\bar{L})}(\|Du_0\| + \bar{C}T)$$

implies (F5). To prove (5.10) we use the monotonicity of the scheme. In particular, for

some j,k with $-p \le j \le q$, $-r \le k \le s+1$ it is

$$\left|\frac{\Delta_+^x v_{j,k}^\rho}{\rho\alpha}(\bar{x},\bar{y})\right| = \left|\frac{v(\bar{x}+\rho(j+1)\alpha,\bar{y}+\rho k\beta) - v(\bar{x}+\rho j\alpha,\bar{y}+\rho k\beta)}{\rho\alpha}\right|$$

But $(\rho(j+1)\alpha,\rho k\beta)$, $(\rho j\alpha,\rho k\beta) \in B_2(0,\rho r)$, therefore by (5.9) it is

$$|\Delta_+^x v_{j,k}^\rho(\bar{x},\bar{y})| \le \bar{\bar{L}}\rho\alpha$$

and similarly

$$|\Delta_+^x w_{j,k}^\rho(\bar{x},\bar{y})| \le \bar{\bar{L}}\rho\alpha,\ |\Delta_+^y v_{j',k'}^\rho(\bar{x},\bar{y})|,\ |\Delta_+^y w_{j',k'}^\rho(\bar{x},\bar{y})| \le \bar{\bar{L}}\rho\beta$$

(5.10) then follows from (5.7).

For (F6) observe that the discussion after the statement of Theorem 2.1 implies that, if $u \in C_b^{0,1}(\mathbb{R}^2)$ with $\|Du\| \le \bar{\bar{L}} + 1$, then

$$\|F(t,\rho,u,u) - F(t,\rho,u,0)\| \le \|u\|$$

Therefore

$$\|F(t,\rho,u,u)\| \le \|u\| + \|F(t,\rho,u,0)\|$$

But

$$|F(t,\rho,u,0)(x,y)| = |-\rho g(t,x,y,u(x,y),0\ldots0,0\ldots0)| \le \rho(\bar{L}\|u\| + C)$$

and thus

$$\|F(t,\rho,u,u)\| \le e^{\rho\bar{L}}(\|u\| + \rho C)\ .$$

Next, for $u \in C_b^{0,1}(\mathbb{R}^2)$ such that $\|u\| \le e^{T\bar{L}}(\|u\| + TC)$ and $\|Du\| \le \bar{\bar{L}}$, let $\bar{u} : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\bar{u}(x,y) = u(x+\eta, y+\xi)$$

for some $(\eta,\xi) \in \mathbb{R}^2$. We have

$$|F(t,\rho,u,u)(x,y) - F(t,\rho,u,u)(x+\eta,y+\xi)| \le \|F(t,\rho,u,u) - F(t,\rho,u,\bar{u})\| +$$

$$+ \rho\left|g(t,x,y,u(x,y), \frac{(\Delta_+^x u)^\rho}{\rho\alpha}(x+\eta,y+\xi), \frac{(\Delta_+^y u)^\rho}{\rho\beta}(x+\eta,y+\xi)) -\right.$$

$$\left.- g(t,x+\eta,y+\xi,u(x+\eta,y+\xi), \frac{(\Delta_+^x u)^\rho}{\rho\alpha}(x+\eta,y+\xi), \frac{(\Delta_+^y u)^\rho}{\rho\beta}(x+\eta,y+\xi))\right|$$

therefore

$$|F(t,\rho,u,u)(x,y) - F(t,\rho,u,u)(x+\eta,y+\xi)| \le$$

$$\le \|Du\| |(\eta,\xi)| + \rho\bar{L}\|Du\| |(\eta,\xi)| + \rho\bar{C}(1+\|Du\|)|(\eta,\xi)|$$

which implies

$$\|DF(t,\rho,u,u)\| \le e^{\rho(\bar{L}+\bar{C})}(\|Du\| + \bar{C}\rho)$$

Since

$$e^{T(\bar{L}+\bar{C})}(\|Du_0\| + \bar{C}T) \le \bar{\bar{L}}$$

(F7) holds.

Finally, for $u \in C_b^{0,1}(\mathbb{R}^N)$ and $\phi \in C_b^2(\mathbb{R}^N)$, if $(x,y) \in \mathbb{R}^2$ is such that

$$|D\phi(x,y)| \le \bar{\bar{L}} + 1$$

then, for $\rho > 0$,

$$\left|\frac{F(t,\rho,u,\phi)(x,y) - \phi(x,y)}{\rho} + H(t,x,y,u(x,y),\phi_x(x,y),\phi_y(x,y))\right|$$

$$= \left|g\left(t,x,y,u(x,y),\frac{(\Delta_+^x\phi)^\rho}{\rho\alpha}(x,y),\frac{(\Delta_+^y\phi)^\rho}{\rho\alpha}(x,y)\right) -\right.$$

$$\left. - g(t,x,y,u(x,y),\phi_x(x,y),\ldots,\phi_x(x,y),\phi_y(x,y),\ldots,\phi_y(x,y))\right|$$

therefore

$$\left|\frac{F(t,\rho,u,\phi)(x,y) - \phi(x,y)}{\rho} + H(t,x,y,u(x,y),\phi_x(x,y),\phi_y(x,y))\right| \le \bar{M}\|D^2\phi\|\rho$$

where $\bar{M} = 2AM_{\max(\|u\|,\bar{\bar{L}}+1)}r$ and thus (F8).

The second part of this section is devoted to the convergence of certain fully implicit finite difference schemes to the viscosity solution of (0.1). We now describe the class of difference schemes to be considered here. For notational simplicity only, we will assume $N = 2$. The definitions and results for general N will be clear from this special case and we will not state them. A generic point in $\mathbb{R}^2$ will be denoted by (x,y) and we will write $Du = (u_x,u_y)$. Let $\alpha,\beta > 0$ be some given positive numbers. For $\rho > 0$, $u : \mathbb{R}^2 \to \mathbb{R}$ we define $\Delta^{x,\pm,\rho}u : \mathbb{R}^2 \to \mathbb{R}$ and $\Delta^{y,\pm,\rho}u : \mathbb{R}^2 \to \mathbb{R}$ by

$$
(5.11)\quad \begin{cases} \Delta^{x,+,\rho}u(x,y) = u(x+\alpha\rho,y) - u(x,y) \\ \Delta^{x,-,\rho}u(x,y) = u(x,y) - u(x-\alpha\rho,y) \\ \Delta^{y,+,\rho}u(x,y) = u(x,y+\rho\beta) - u(x,y) \\ \Delta^{y,-,\rho}u(x,y) = u(x,y) - u(x,y-\rho\beta) \end{cases}
$$

If $u \in C^{0,1}(\mathbb{R}^2)$, it is easy to see that, for every $(x,y) \in \mathbb{R}^2$, it is

$$
(5.12)\qquad \frac{|\Delta^{x,\pm,\rho}u(x,y)|}{\rho\alpha},\ \frac{|\Delta^{y,\pm,\rho}u(x,y)|}{\rho\beta} \le \|Du\|
$$

As far as $H : [0,T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is concerned, here we assume that it satisfies (H1), (H2),

$$
(H8)\quad \begin{cases} \text{There is a constant } L > 0 \text{ such that} \\ \quad |H(t,(x,y),r,(p,q)) - H(\bar{t},(\bar{x},\bar{y}),\bar{r},(\bar{p},\bar{q}))| \le \\ \qquad \le L(|t-\bar{t}| + |(x,y)-(\bar{x},\bar{y})| + |r-\bar{r}| + |(p,q)-(\bar{p},\bar{q})|) \\ \text{for every } t,\bar{t} \in [0,T],\ r,\bar{r} \in \mathbb{R} \text{ and } (x,y),(\bar{x},\bar{y}),(p,q),(\bar{p},\bar{q}) \in \mathbb{R}^2 \end{cases}
$$

and

$$
(H9)\quad \begin{cases} H \text{ is monotone with respect to } p \text{ and } q \text{ for} \\ \text{every } t \in [0,T],\ r \in \mathbb{R} \text{ and } (x,y),(p,q) \in \mathbb{R}^2 \end{cases}
$$

For $u,v,w \in BUC(\mathbb{R}^2)$ and $0 < \rho \le t \le T$ let $T(t,\rho,u,v)w : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$
(5.12)\quad T(t,\rho,u,v)w(x,y) = v(x,y) - \rho H(t,x,y,u(x,y), \frac{\Delta^{\pm,x,\rho}w}{\rho\alpha}(x,y), \frac{\Delta^{\pm,y,\rho}w}{\rho\beta}(x,y))
$$

where we use $\Delta^{+,x,\rho}$ ($\Delta^{+,y,\rho}$), if H is nonincreasing with respect to $p(q)$, and $\Delta^{-,x,\rho}$ ($\Delta^{-,y,\rho}$), if H is nondecreasing with respect to $p(q)$. In view of (5.11), (H1), (H2) and (H8), it is obvious that $T(t,\rho,u,v)w \in BUC(\mathbb{R}^2)$. Moreover, we have

Lemma 4.1. For α,β sufficiently large, $T(t,\rho,u,v)$ has a fixed point in $BUC(\mathbb{R}^2)$. If, moreover, $u,v \in C_b^{0,1}(\mathbb{R}^2)$, then the fixed point is in $C_b^{0,1}(\mathbb{R}^2)$.

Proof. We first show that, for α,β sufficiently large, $T(t,\rho,u,v)$ is a strict contraction in the $\| \ \|$-norm. Indeed, if $w,z \in BUC(\mathbb{R}^2)$, then

$$|T(t,\rho,u,v)w(x,y) - T(t,\rho,u,v)z(x,y)| \le$$

$$\le \rho|H(t,x,y,u(x,y), \frac{\Delta^{t,x,\rho}w}{\rho\alpha}(x,y), \frac{\Delta^{t,y,\rho}w}{\rho\beta}(x,y)) -$$

$$- H(t,x,y,u(x,y), \frac{\Delta^{t,x,\rho}z}{\rho\alpha}(x,y), \frac{\Delta^{t,y,\rho}z}{\rho\beta}(x,y))| \le 2\sqrt{2}\, L(\frac{1}{\alpha} + \frac{1}{\beta})\|w - z\| .$$

So, if $C_0 = 2\sqrt{2}\, L(\frac{1}{\alpha} + \frac{1}{\beta}) < \frac{1}{2}$, we have

$$\|T(t,\rho,u,v)w - T(t,\rho,u,v)z\| \le C_0\|w - z\| \tag{5.13}$$

By the contraction mapping principle, (5.13) implies the existence of a unique fixed point of $T(t,\rho,u,v)$ in $BUC(\mathbb{R}^2)$.

If $u,v,w \in C_b^{0,1}(\mathbb{R}^2)$, it follows directly from (H8) and (5.12) that

$$\|DT(t,\rho,u,v)w\| \le \|Dv\| + \rho L(1 + \|Du\|) + C_0\|Dw\| \tag{5.14}$$

So, if w is such that

$$\|Dw\| \le \frac{\|Dv\| + \rho L(1 + \|Du\|) + C_0}{1 - C_0}$$

(5.14) implies

$$\|DT(t,\rho,u,v)w\| \le \frac{\|Dv\| + \rho L(1 + \|Du\|) + C_0}{1 - C_0} \tag{5.15}$$

In view of (5.13), it follows that $T(t,\rho,u,v) : C_b^{0,1}(\mathbb{R}^2) \to C_b^{0,1}(\mathbb{R}^2)$ has unique fixed point $\tilde{w} \in C_b^{0,1}(\mathbb{R}^2)$, which satisfies

$$\left\{ \begin{array}{l} \|D\tilde{w}\| \le \dfrac{\|Dv\| + \rho L(1 + \|Du\|) + C_0}{1 - C_0} \\ \text{and} \\ \|\tilde{w}\| \le \dfrac{\|v\| + \rho L(C + \|u\|) + C_0}{1 - C_0} \end{array} \right. \tag{5.16}$$

where the second inequality follows from (H2) (C is the constant in (H2)), (H8) and

$$\|T(t,\rho,u,v)w\| \leq \|v\| + \rho(C + \|u\|) + C_0\|w\| \tag{5.17}$$

and it is valid even when $u,v,w \in BUC(\mathbb{R}^2)$

Next for $0 \leq \rho \leq t \leq T$ let $F(t,\rho,\cdot,\cdot) : BUC(\mathbb{R}^2) \times BUC(\mathbb{R}^2) \to BUC(\mathbb{R}^2)$ be defined by

$$\begin{cases} \text{If } \rho = 0, \text{ then } F(t,\rho,u,v) = v \\ \text{If } \rho > 0, \text{ then } F(t,\rho,u,v) \text{ is the unique} \\ \text{fixed point of } T(t,\rho,u,v) : BUC(\mathbb{R}^2) \to BUC(\mathbb{R}^2) \end{cases} \tag{5.18}$$

The theorem is:

<u>Theorem 5.2</u>. (a) Let $H : [0,T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy (H1), (H2), (H8) and (H9) and $u_0 \in BUC(\mathbb{R}^N)$. For a partition P of $[0,T]$, define $u_P : \bar{Q}_T \to \mathbb{R}$ by (2.1) and (5.18). Let u be the viscosity solution of (0.1). Then

$$\|u_P - u\| \to 0 \quad \text{as} \quad |P| \to 0 . \tag{5.19}$$

(b) If $u_0 \in C_b^{0,1}(\mathbb{R}^2)$, then, for sufficiently small $|P|$,

$$\|u_P - u\| \leq K|P|^{1/2}$$

where K is a constant which depends only on $\|u_0\|$, $\|Du_0\|$.

<u>Proof</u>. (a) It suffices to check the assumptions of Theorem 2.1 (b). (F1) is satisfied because of (5.18). Moreover, (F3) is an immediate consequence of the definition of $F(t,\rho,u,v)$ and $T(t,\rho,u,v)$. To check (F4) observe that, in view of (5.16), we have

$$\|F(t,\rho,u,u) - u\| \leq \rho \sup_{\substack{((x,y),\tau)\in\bar{Q}_T \\ |(p,q)|\leq \frac{\|Du\|+TL(1+\|Du\|)+C_0}{1-C_0}}} |H(\tau,x,y,u(x,y),p,q)|$$

The continuity of $(t,\rho) \to F(t,\rho,u,u)$ for $u \in C_b^{0,1}(\mathbb{R}^2)$ follows from the above inequality for $\rho = 0$ and from the properties of $T(t,\rho,u,u)$ and (5.15), (5.16), (5.17), in the case that $\rho > 0$.

For (F9) we need to specify the monotonicity of H. In particular, here we are going to assume that H is nonincreasing with respect to p and nondecreasing with respect to q. If another combination is true, then one has to modify what follows in an appropriate

way. If $\rho = 0$, then

$$\|F(t,0,u,v) - F(t,0,u,\bar{v})\| = \|v - \bar{v}\|$$

If $\rho > 0$, then $w = F(t,\rho,u,v)$ and $\bar{w} = F(t,\rho,u,\bar{v})$ satisfy

$$(5.19) \qquad \begin{cases} w(x,y) + \rho H\left(t,x,y,u(x,y), \dfrac{\Delta^{+,x,\rho}w}{\rho\alpha}(x,y), \dfrac{\Delta^{-,y,\rho}w}{\rho\beta}(x,y)\right) = v(x,y) \\ \\ \bar{w}(x,y) + \rho H\left(t,x,y,\bar{u}(x,y), \dfrac{\Delta^{+,x,\rho}\bar{w}}{\rho\alpha}(x,y), \dfrac{\Delta^{-,y,\rho}\bar{w}}{\rho\beta}(x,y)\right) = \bar{v}(x,y) \end{cases}$$

We are going to show that

$$(5.20) \qquad \sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ \leq \|v - \bar{v}\| + \rho L\|u - \bar{u}\|$$

The above, together with a similar inequality for $\sup_{(x,y)\in\mathbb{R}^2} (\bar{w}(x,y) - w(x,y))^-$, which is proved exactly as (5.20), implies (F9). To this end, observe that, if

$$\sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ = 0$$

then there is nothing to show. Without any loss of generality, we may assume

$$(5.21) \qquad \sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ > 0 \ .$$

In this case let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\Phi(x,y) = (w(x,y) - \bar{w}(x,y))^+$$

Since Φ is bounded, for every $\delta > 0$ there is a $(x_1,y_1) \in \mathbb{R}^2$ such that

$$\Phi(x_1,y_1) > \sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ - \delta$$

Next choose $\zeta \in C_0^2(\mathbb{R}^2)$ such that $0 \leq \zeta \leq 1$, $|D\zeta| \leq 1$, $\zeta(x_1,y_1) = 1$ and define $\Psi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Psi(x,y) = \Phi(x,y) + 2\delta\zeta(x,y)$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1) = \Phi(x_1,y_1) + 2\delta > \sup_{(x,y)\in\mathbb{R}^2} \Phi(x,y) + \delta$$

there is a point $(x_0,y_0) \in \mathbb{R}^2$ such that

(5.22) $$\Psi(x_0,y_0) \geq \Psi(x,y) \quad \text{for every } (x,y) \in \mathbb{R}^2 .$$

Moreover, it is easy to see that, for $\delta < \frac{1}{2} \sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+$,

(5.23) $$\begin{cases} w(x_0,y_0) - \bar{w}(x_0,y_0) > 0 \\ \text{and} \\ \sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ \leq (w(x_0,y_0) - \bar{w}(x_0,y_0))^+ + 2\delta \end{cases}$$

Using (5.19) and (5.23) we have

$$\sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ \leq \|v - \bar{v}\| + 2\delta +$$

$$+ \rho H(t,x_0,y_0,\bar{u}(x_0,y_0), \frac{\Delta^{+,x,\rho}\bar{w}}{\rho\alpha}(x_0,y_0), \frac{\Delta^{-,y,\rho}\bar{w}}{\rho\beta}(x_0,y_0)) -$$

$$- \rho H(t,x_0,y_0,u(x_0,y_0), \frac{\Delta^{+,x,\rho}w}{\rho\alpha}(x_0,y_0), \frac{\Delta^{-,y,\rho}w}{\rho\beta}(x_0,y_0))$$

therefore

$$\sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ \leq \|v - \bar{v}\| + 2\delta + \rho L\|u - \bar{u}\| +$$

$$+ \rho H(t,x_0,y_0,u(x_0,y_0), \frac{\Delta^{+,x,\rho}\bar{w}}{\rho\alpha}(x_0,y_0), \frac{\Delta^{-,y,\rho}\bar{w}}{\rho\beta}(x_0,y_0)) -$$

$$- \rho H(t,x_0,y_0,u(x_0,y_0), \frac{\Delta^{+,x,\rho}w}{\rho\alpha}(x_0,y_0), \frac{\Delta^{-,y,\rho}w}{\rho\beta}(x_0,y_0))$$

But, in view of (5.22) and (5.23), it is

$$\frac{\Delta^{+,x,\rho}w}{\rho\alpha}(x_0,y_0) \leq \frac{\Delta^{+,x,\rho}\bar{w}}{\rho\alpha}(x_0,y_0) - 2\delta\,\frac{\Delta^{+,x,\rho}\zeta}{\rho\alpha}(x_0,y_0)$$

and

$$\frac{\Delta^{-,y,\rho}w}{\rho\beta}(x_0,y_0) \geq \frac{\Delta^{-,y,\rho}\bar{w}}{\rho\beta}(x_0,y_0) - 2\delta\,\frac{\Delta^{-,y,\rho}\zeta}{\rho\beta}(x_0,y_0)$$

Thus, by the monotonicity of H, we have

$$H(t,x_0,y_0,u(x_0,y_0), \frac{\Delta^{+,x,\rho}w}{\rho\alpha}(x_0,y_0), \frac{\Delta^{-,y,\rho}w}{\rho\beta}(x_0,y_0)) -$$

$$- H(t,x_0,y_0,u(x_0,y_0), \frac{\Delta^{+,x,\rho}\bar{w}}{\rho\alpha}(x_0,y_0) +$$

$$- 2\delta \frac{\Delta^{+,x,\rho}\zeta}{\rho\alpha}(x_0,y_0), \frac{\Delta^{-,x,\rho}\bar{w}}{\rho\beta}(x_0,y_0) - 2\delta \frac{\Delta^{-,y,\rho}\zeta}{\rho\beta}(x_0,y_0)) > 0$$

The above inequality, together with (5.24) and (H9), implies

$$\sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \bar{w}(x,y))^+ < \|v - \bar{v}\| + \rho L\|u - \bar{u}\| + 2\delta C_1$$

where C_1 is a constant which depends on L,α,β. Letting $\delta \downarrow 0$ we obtain the result.

Next we check (F10). In view of (F9), we have

$$\|F(t,\rho,u,u)\| < \|F(t,\rho,u,u) - F(t,\rho,0,0)\| + \|F(t,\rho,0,0)\| <$$

$$< (1 + \rho L)\|u\| + \|F(t,\rho,0,0)\|$$

Moreover, if $w = F(t,\rho,0,0)$, then

$$w(x,y) = -\rho H(t,(x,y),0, \frac{\Delta^{+,x,\rho}w}{\rho\alpha}(x,y), \frac{\Delta^{-,y,\rho}}{\rho\beta}(x,y))$$

But (5.16) and (5.18) imply

$$\|Dw\| < \frac{TL + C_0}{1 - C_0}$$

thus

$$\|F(t,\rho,0,0)\| < \rho \sup_{\substack{(x,y)\in\mathbb{R}^2 \\ \tau\in[0,T] \\ |(p,q)|<\frac{TL+C_0}{1-C_0}}} |H(\tau,(x,y),0,p,q)|$$

and

$$\|F(t,\rho,u,u)\| < e^{\rho L}(\|u\| + \rho C_7)$$

where

$$C_7 = \sup_{\substack{((x,y),\tau)\in\bar{Q}_T \\ |(p,q)|\le\frac{TL+C_0}{1-C_0}}} |H(\tau,x,y,0,p,q)|$$

For (F11) observe that, if $u \in C_b^{0,1}(\mathbb{R}^2)$, then $F(t,\rho,u,u) \in C_b^{0,1}(\mathbb{R}^2)$ by Lemma 5.1 and (5.18). Let $(\xi,\eta) \in \mathbb{R}^2$. If $w = F(t,\rho,u,u)$, let $\bar{w} : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\bar{w}(x,y) = w(x+\xi, y+\eta) .$$

It is easy to check that

$$\bar{w} = F(t,\rho,\bar{u},\bar{u} + \rho f)$$

where

$$\bar{u}(x,y) = u(x+\xi, y+\eta)$$

and

$$f(x,y) = H(t,x,y,\bar{u}(x,y), \frac{\Delta^{+,x,\rho}\bar{w}}{\rho\alpha}(x,y), \frac{\Delta^{-,y,\rho}\bar{w}}{\rho\beta}(x,y)) -$$

$$- H(t,x+\xi,y+\eta,\bar{u}(x,y), \frac{\Delta^{+,x,\rho}\bar{w}}{\rho\alpha}(x,y), \frac{\Delta^{-,y,\rho}\bar{w}}{\rho\beta}(x,y))$$

But then (F9) implies

$$\|w - \bar{w}\| \le \|u - (\bar{u} + \rho f)\| + \rho L\|u - \bar{u}\|$$

So, in view of (H8),

$$\|Dw\| \le \|Du\| + \rho L + \rho L\|Du\| \le e^{\rho L}(\|Du\| + \rho L)$$

and thus (F11).

Finally, we want to verify (F12). We are going to show that

$$(5.24)^{\pm} \qquad \sup_{(x,y)\in\mathbb{R}^2} (F(t,\rho,u,\phi)(x,y) - \phi(x,y) + \rho H(t,x,y,u(x,y),D\phi(x,y)))^{\pm} \le$$

$$\le C_5(1 + \|D^2\phi\| + \|D\phi\|)\rho^2$$

where $C_5 = C_5(\|Du\|)$. Here we prove only $(5.24)^+$, since $(5.24)^-$ can be shown in exactly the same way. Let $w = F(t,\rho,u,\phi)$. Without any loss of generality, we may assume that

$$\sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \phi(x,y) + \rho H(t,x,y,u(x,y),D\phi(x,y)))^+ > 0$$

In this case let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\Phi(x,y) = (w(x,y) - \phi(x,y) + \rho H(t,x,y,u(x,y),D\phi(x,y)))^+$$

Since Φ is bounded, for every $\delta > 0$ there is a point $(x_1,y_1) \in \mathbb{R}^2$ such that

$$\Phi(x_1,y_1) > \sup_{(x,y)\in\mathbb{R}^2} \Phi(x,y) - \delta$$

Next choose $\zeta \in C_0^\infty(\mathbb{R}^2)$ such that $0 \le \zeta \le 1$, $|D\zeta| \le 1$, $|\Delta\zeta| \le 1$, $\zeta(x_1,y_1) = 1$ and define $\Psi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Psi(x,y) = \Phi(x,y) + 2\delta\zeta(x,y) .$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1) = \Phi(x_1,y_1) + 2\delta > \sup_{(x,y)\in\mathbb{R}^2} \Phi(x,y) + \delta$$

there is a point $(x_0,y_0) \in \mathbb{R}^2$ such that

(5.25) $$\Psi(x_0,y_0) \ge \Psi(x,y) \quad \text{for every } (x,y) \in \mathbb{R}^2 .$$

It is easy to check that if

$$\delta < \frac{1}{2} \sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \phi(x,y) + \rho H(t,x,y,u(x,y),D\phi(x,y)))^+$$

then

(5.26) $$\begin{cases} w(x_0,y_0) - \phi(x_0,y_0) + \rho H(t,x_0,y_0,u(x_0,y_0),D\phi(x_0,y_0)) > 0 \\ \text{and} \\ \sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \phi(x,y) + \rho H(t,x,y,u(x,y),D\phi(x,y)))^+ \le \\ \quad \le (w(x_0,y_0) - \phi(x_0,y_0) + \rho H(t,x_0,y_0,u(x_0,y_0),D\phi(x_0,y_0)) + 2\delta \end{cases}$$

In this case we have

(5.27) $$\begin{aligned} w(x_0,y_0) - \phi(x_0,y_0) + \rho H(t,x_0,y_0,u(x_0,y_0),D\phi(x_0,y_0)) = \\ = \rho[H(t,x_0,y_0,u(x_0,y_0),\phi_x(x_0,y_0),\phi_y(x_0,y_0)) - \\ - H(t,x_0,y_0,u(x_0,y_0), \frac{\Delta^{+,x,\rho}w}{\rho\alpha}(x_0,y_0), \frac{\Delta^{-,y,\rho}w}{\rho\beta}(x_0,y_0))] \end{aligned}$$

But, in view of (5.25), it is

$$\frac{\Delta^{+,x,\rho}w}{\rho\alpha}(x_0,y_0) \le \frac{1}{\rho\alpha}\Delta^{+,x,\rho}(\phi - \rho H(t,\cdot,\cdot,u(\cdot,\cdot),D\phi(\cdot,\cdot)) - 2\delta\zeta(\cdot,\cdot))(x_0,y_0)$$

and

$$\frac{\Delta^{-,y,\rho}w}{\rho\beta}(x_0,y_0) \ge \frac{1}{\rho\beta}\Delta^{-,y,\rho}(\phi - \rho H(t,\cdot,\cdot,u(\cdot,\cdot),D\phi(\cdot,\cdot)) - 2\delta\zeta(\cdot,\cdot))(x_0,y_0)$$

The above inequalities together with (H8), the monotonicity of H, (5.12), (5.26) and (5.27) imply

$$\sup_{(x,y)\in\mathbb{R}^2} (w(x,y) - \phi(x,y) + \rho H(t,x,y,u(x,y),D\phi(x,y))^+ \le$$

$$\le 2\delta + \rho[H(t,x_0,y_0,u(x_0,y_0),\phi_x(x_0,y_0),\phi_y(x_0,y_0)) -$$

$$- H(t,x_0,y_0,u(x_0,y_0), \frac{1}{\rho\alpha}\Delta^{+,x,\rho}(\phi - \rho H(t,\cdot,\cdot,u(\cdot,\cdot),D\phi(\cdot,\cdot)) -$$

$$- 2\delta\zeta)(x_0,y_0), \frac{1}{\rho\beta}\Delta^{-,y,\rho}(\phi - \rho H(t,\cdot,\cdot,u,D\phi) - 2\delta\zeta)(x_0,y_0))] \le$$

$$\le 2\delta + \sqrt{2}\,(\max(\alpha,\beta)\|D^2\phi\| + L(1 + \|Du\| + \|D^2\phi\|) + 2\delta)L\rho^2 .$$

Letting $\delta \downarrow 0$ we obtain $(5.24)^+$.

(b) It follows from Theorem 2.1 (a), since in part (a) above we checked all of its hypotheses.

<u>Remark 5.1</u>. One can prove the same result in the case that H satisfy (H4) type assumptions.

<u>Remark 5.2</u>. Assumption (H9) is not really restrictive. In particular, it is easy to check that, if $u \in BUC(\bar{Q}_T)$ is the viscosity solution of

$$\begin{cases} u_t + H(t,x,u,Du) = 0 & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

then $v(x,t) = u(x - tc,t)$, where $c \in \mathbb{R}^N$, is the viscosity solution of

$$\begin{cases} v_t + H(t,x - tc,Dv) + c\cdot Dv = 0 & \text{in } \Omega_T \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

where $c\cdot Dv$ denotes the usual inner product $\mathbb{R}^N$. In view of (H9), we see that, with an appropriate choice of c, we can always achieve (H10).

SECTION 6

We begin by introducing some notation. In particular, if for $u_0 \in BUC(\mathbb{R}^N)$ and $0 \le s \le T$

$$(6.1)_s \qquad \begin{cases} u_t + H(t,x,u,Du) = 0 & \text{in } \mathbb{R}^N \times (s,T] \\ u(x,s) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

has a unique viscosity solution $u \in BUC(\mathbb{R}^N \times [s,T])$, then we write

$$(6.2) \qquad U(t,s)u_0 = u(\cdot,t)$$

Similarly, if for $u_0, w \in BUC(\mathbb{R}^N)$ and $0 \le s \le T$

$$(6.3)_s \qquad \begin{cases} u_t + H(t,x,w(x),Du) = 0 & \text{in } \mathbb{R}^N \times (s,T] \\ u(x,s) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

has a unique viscosity solution $u \in BUC(\mathbb{R}^N \times [s,T])$, we write

$$(6.4) \qquad U(t,s,w)u_0 = u(\cdot,t) \; .$$

Moreover, if for $\lambda > 0$, $t \in [0,T]$ and $v \in BUC(\mathbb{R}^N)$

$$(6.5) \qquad u + \lambda H(t,x,u,Du) = v \quad \text{in } \mathbb{R}^N$$

has a unique viscosity solution $u \in BUC(\mathbb{R}^N)$, we write

$$(6.6) \qquad J(t,\lambda)v = u$$

Finally, if for $\lambda > 0$, $t \in [0,T]$ and $v,w \in BUC(\mathbb{R}^N)$ the problem

$$u + \lambda H(t,x,w,Du) = v \quad \text{in } \mathbb{R}^N$$

has a unique viscosity solution $u \in BUC(\mathbb{R}^N)$, we write

$$J(t,\lambda,w)v = u \; .$$

The first theorem of this section is

Theorem 6.1. *(a)* For $i = 1,2$ let $H_i : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1), (H2), (H4) (with constant C_i independent of R) and (H3) (with constant $\gamma_i \le 0$ independent of R). For $u_0 \in BUC(\mathbb{R}^N)$ let $u \in BUC(\bar{Q}_T)$ be the viscosity solution of (0.1) in $\bar{Q}_T$ with $H = H_1 + H_2$. If, for a partition P of $[0,T]$, $u_P : \bar{Q}_T \to \mathbb{R}$ is defined by (2.42) using $F(t,\rho,v) = U_2(t,t-\rho)U_1(t,t-\rho)v$, then

$$(6.9) \qquad \|u_P - u\| \to 0 \quad \text{as} \quad |P| \to 0$$

(b) Suppose that, for $i = 1,2$, H_i satisfies (H1), (H2), (H4), (H5), (H6), and (H7) with constants $\bar{L}_i, N_i, M_i$ independent of R. If $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ and $u_P : \bar{Q}_T \to \mathbb{R}$ is defined by either (2.1) using $F(t,\rho,w,v) = U_2(t,t-\rho,w)U_1(t,t-\rho,w)$ or (2.42) using $F(t,\rho,v) = U_2(t,t-\rho)U_1(t,t-\rho)v$, then there exists a constant K depending only on $\|u_0\|$ and $\|Du_0\|$ such that

(6.10) $$\|u_P - u\| \le K|P|^{1/2}$$

for $|P|$ sufficiently small.

Remark 6.1. The assumption, that H_1, H_2 satisfy (H3) or (H4) and (H7) with constants independent of R, is made only for simplicity. In fact, one can always reduce to this case by using Proposition 1.5, truncating H_1, H_2 in an appropriate way and restricting, if necessary, T.

Proof of Theorem 6.1. (b) We first prove (6.10) in the case that u_P is defined by (2.1) for

(6.11) $$F(t,\rho,w,v) = U_2(t,t-\rho,w)U_1(t,t-\rho,w)v$$

To this end, it suffices to check the assumptions of Theorem 2.1 (a). In view of Proposition 1.9, if $w,v \in C_b^{0,1}(\mathbb{R}^N)$, then $F(t,\rho,w,v) \in C_b^{0,1}(\mathbb{R}^N)$. Moreover, since for $i = 1,2$, $U_i(s,s,w)v = v$ for every $w,v \in C_b^{0,1}(\mathbb{R}^N)$, it is immediate that (F1) is satisfied. (F2) follows from the fact that, for $i = 1,2$ and $u,v,\bar{v} \in C_b^{0,1}(\mathbb{R}^N)$, the viscosity solution of

$$\begin{cases} \dfrac{\partial u_i}{\partial \tau} + H_i(\tau,x,u(x),Du_i) = 0 & \text{in } Q_T \\ u_i(x,0) = v & \text{in } \mathbb{R}^N \end{cases}$$

is Lipschitz continuous with respect to τ and, moreover,

$$\|U_i(t,s,u)v - U_i(t,s,u)\bar{v}\| \le \|v - \bar{v}\|$$

(F3) is an immediate consequence of the definition of the viscosity solution. Next, and in view of (1.11), for $u,v,\bar{v} \in C_b^{0,1}(\mathbb{R}^N)$ it is

$$\|F(t,\rho,u,v) - F(t,\rho,u,\bar{v})\| \le \|v - \bar{v}\|$$

which, by the discussion after the statement of Theorem 2.1, implies (F5) for $\bar{\bar{L}} = \infty$. Moreover, in view of (1.12), for $u \in C_b^{0,1}(\mathbb{R}^N)$ we have

$$\|F(t,\rho,u,u)\| \le \|U_1(t,t-\rho,u)u\| + \rho(C_2 + \bar{L}_2\|u\|) \le$$
$$\le \|u\| + \rho(\bar{L}_1 + \bar{L}_2)\|u\| + \rho(C_1 + C_2)$$

and thus (F6), where for $i = 1,2$, C_i is given by (H2) and $\bar{L}_i$ is given by (H5).

For $R = e^{T(\bar{L}_1+\bar{L}_2)}(\|u_0\| + T(C_1 + C_2))$ and $i = 1,2$, let $\bar{C}_i = C_R^i$, where C_R^i is given by (H4). In view of (1.13), we have

$$\|DU_1(t,\rho,u,u)\| \le e^{(2\bar{C}_1+\bar{L}_1)\rho}(\|Du\| + \rho\bar{C}_1)$$

and

$$\|DF(t,\rho,u,u)\| \le e^{(2\bar{C}_2+\bar{L}_2)\rho}(e^{(2\bar{C}_1+\bar{L}_1)\rho}(\|Du\| + \rho\bar{C}_1) + \rho\bar{C}_2)$$

i.e.

$$\|DF(t,\rho,u,u)\| \le e^{[2(\bar{C}_1+\bar{C}_2)+(\bar{L}_1+\bar{L}_2)]\rho}(\|Du\| + \rho(\bar{C}_1 + \bar{C}_2))$$

which proves (F7). Proposition 1.5 also proves (F4), since (1.16) implies that

$$\|F(t,\rho,u,u) - u\| \le$$
$$\le \rho\Big(\sup_{\substack{(x,\tau)\in\mathbb{R}^N\times[t-\rho,t]\\ |p|\le\|DU_1(t,t-\rho,u)u\|}} |H_2(\tau,x,u(x),p)| + \sup_{\substack{(x,\tau)\in\mathbb{R}^N\times[t-\rho,t]\\ |p|\le\|Du\|}} |H_1(\tau,x,u(x),p)|\Big)$$

Finally, we need to check (F8). To this end, for $i = 1,2$, consider smooth functions $\tilde{H}_i^n : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ which satisfy the same conditions as H_i, with constants depending on the constants of H_i, $\|u\|$, $\|Du\|$ and are such that

$$\begin{cases} \tilde{H}_i^n(t,x,p) \to H_i(t,x,u(x),p) \quad \text{as} \quad n \to \infty \\ \text{uniformly on} \quad [0,T] \times \mathbb{R}^N \times B_N(0,R) \quad \text{for each} \quad R > 0. \end{cases}$$

Then, in view of the previous discussion and Proposition 1.5, $\tilde{U}_2^n(t,t-\rho)\tilde{U}_1(t,t-\rho)\phi \in C_b^{0,1}(\mathbb{R}^N)$ where $\tilde{U}_i^n$ corresponds to $\tilde{H}_i^n$ by (6.2) and $\phi \in C_b^2(\mathbb{R}^N)$. Moreover, there exists a constant $\bar{\bar{C}}$ depending only on $\|u\|$, $\|Du\|$, H_i, ϕ and $\|D\phi\|$ such that

$$(6.12)\quad \begin{cases} \|DU_2(\tau,\tau-\sigma,u)U_1(t,t-\rho,u)\phi\|, \|DU_1(t,t-\rho,u)\phi\| \le \bar{\bar{C}} \\ \text{and} \\ \|D\tilde{U}_2^n(\tau,\tau-\sigma)\tilde{U}_1^n(t,t-\rho)\phi\|, \|D\tilde{U}_1^n(t,t-\rho)\phi\| \le \bar{\bar{C}} \end{cases}$$

for every $(\tau,\sigma),(t,\rho) \in K$. Then, in view of Proposition 1.4, we have

$$(6.13)\quad \|F(t,\rho,u,\phi) - \tilde{U}_2^n(t,t-\rho)\tilde{U}_1^n(t,t-\rho)\phi\| \le \rho \sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le\bar{\bar{C}}}} |\tilde{H}_i^n(\tau,x,p) - H_i(\tau,x,u(x),p)|$$

and therefore

$$(6.14)\quad \left\|\frac{F(t,\rho,u,\phi) - \phi}{\rho} + H_1(t,\cdot,u,D\phi) + H_2(t,\cdot,u,D\phi)\right\| \le$$

$$\le \left\|\frac{\tilde{U}_2^n(t,t-\rho)\tilde{U}_1^n(t,t-\rho)\phi - \phi}{\rho} + (\tilde{H}_1 + \tilde{H}_2)(t,\cdot,D\phi)\right\| +$$

$$+ 2\sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le\bar{\bar{C}}}} |\tilde{H}_i^n(\tau,x,p) - H_i(\tau,x,u(x),p)| .$$

In order to finish we need the following lemma.

<u>Lemma 6.2</u>. (a) Let $H_1 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be smooth and assume that it satisfies (H1), (H2), (H4), (H5), (H6) and (H7) with constants independent of R. If, for $\phi \in C_b^2(\mathbb{R}^N)$, $u_1 \in BUC(\mathbb{R}^N \times [t,\bar{t}])$ is the viscosity solution of

$$\begin{cases} \dfrac{\partial u_1}{\partial \tau} + H_1(\tau,x,Du_1) = 0 & \text{in } \mathbb{R}^N \times (t,\bar{t}] \\ u_1(x,t) = \phi(x) & \text{in } \mathbb{R}^N \end{cases}$$

then for $\tau \in [t,\bar{t}]$

$$\|u_1(\cdot,\tau) - \phi + (\tau - t)H_1(\bar{t},\cdot,D\phi)\| \le \Gamma_1(\tau - t)(\bar{t} - t)(1 + \|D\phi\| + \|D^2\phi\|)$$

where Γ_1 depends only on the constants related with H_1.

(b) Let $H_2 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfy the same hypotheses as H_1 in part (a). If $u_2 \in BUC(\mathbb{R}^N \times [t,\bar{t}])$ is the viscosity solution of

$$\begin{cases} \dfrac{\partial u_2}{\partial \tau} + H_2(\tau,x,Du_2) = 0 & \text{in } \mathbb{R}^N \times (t,\bar{t}] \\ u_2(x,t) = u_1(x,\bar{t}) & \text{in } \mathbb{R}^N \end{cases}$$

then for $\tau \in [t,\bar{t}]$

(6.16) $$\|u_2(\cdot,\tau) - \phi + (\tau - t)H_2(\bar{t},\cdot,D\phi) + (\bar{t} - t)H_1(\bar{t},\cdot,D\phi)\| \le$$
$$\le \Gamma_2(\tau - t)(\bar{t} - t)(1 + \|D\phi\| + \|D^2\phi\|) + \|u_1(\cdot,\bar{t}) - \phi + (\bar{t} - t)H_1(\bar{t},\cdot,D\phi)\|$$

where Γ_2 is a constant, which depends only on the constants related with H_2, H_1 by (H4), (H5), (H6) and (H7).

__Proof__. (a) Here we only show that

(6.17) $$\|(u_1(\cdot,\tau) - \phi + (\tau - t)H_1(\bar{t},\cdot,D\phi))^+\| \le \Gamma_1(\tau - t)(\bar{t} - t)(1 + \|D\phi\| + \|D^2\phi\|)$$

since the other inequality can be proved in exactly the same way. To this end, let $m : [t,\bar{t}] \to \mathbb{R}$ be defined by

(6.18) $$m(\tau) = \sup_{x\in\mathbb{R}^N} (u(x,\tau) - \phi(x) + (\tau - t)H_1(\bar{t},x,D\phi(x)))^+$$

We claim that there is a constant Γ_1 such that m is the viscosity solution of

(6.19) $$m'(\tau) \le \Gamma_1(1 + \|D\phi\| + \|D^2\phi\|)$$

and therefore by *Proposition 1.1 (b) the result. To prove this claim,* let $\eta \in C^\infty((t,\bar{t}))$ and assume that $\bar{\tau} \in (t,\bar{t})$ is a strict local maximum of $m - \eta$ on $I = [\bar{\tau} - \alpha,\bar{\tau} + \alpha]$ $(t,\bar{t})$ for some $\alpha > 0$. We want to have

(6.20) $$\eta'(\tau) \le \Gamma_1(1 + \|D\phi\| + \|D^2\phi\|)$$

If $m(\bar{\tau}) = 0$, then $\bar{\tau}$ is a local minimum of η, therefore $\eta'(\bar{\tau}) = 0$ and (6.21) is satisfied. If $m(\bar{\tau}) > 0$, let $\Phi : \mathbb{R}^N \times I \to \mathbb{R}$ be defined by

$$\Phi(x,\tau) = (u(x,\tau) - \phi(x) + (\tau - t)H_1(\bar{t},x,D\phi(x)))^+ - \eta(\tau)$$

Since Φ is bounded, for every $\delta > 0$, there is a point $(x_1,\tau_1) \in \mathbb{R}^N \times I$, such that

$$\Phi(x_1,\tau_1) > \sup_{(x,\tau)\in\mathbb{R}^N\times[t,\bar{t}]} \Phi(x,\tau) - \delta$$

Next choose $\zeta \in C_0^\infty(\mathbb{R}^N)$ so that $0 \le \zeta \le 1$, $|D\zeta| \le 1$, $|D^2\zeta| \le 2$, $\zeta(x_1) = 1$ and define $\Psi : \mathbb{R}^N \times I \to \mathbb{R}$ by

(6.21) $$\Psi(x,\tau) = \Phi(x,\tau) + 2\delta\zeta(x)$$

Since $\Psi = \phi$ off the support of ζ and

$$\Psi(x_1,\tau_1) > \sup_{(x,\tau)\in R^N\times I} \phi(x,\tau) + \delta$$

there is a point $(x_0,\tau_0) \in R^N \times I$ such that

$$\Psi(x_0,\tau_0) > \Psi(x,\tau) \quad \text{for every} \quad (x,\tau) \in R^N \times I \tag{6.22}$$

We claim that (x_0,τ_0) satisfies

$$\begin{cases} \text{As} \quad \delta \downarrow 0, \quad \tau_0 \to \bar{\tau} \quad \text{and} \\ (u(x_0,\tau_0) - \phi(x_0) + (\tau_0 - t)H_1(\bar{t},x_0,D\phi(x_0)))^+ = \\ \qquad = u(x_0,\tau_0) - \phi(x_0) + (\tau_0 - t)H_1(\bar{t},x_0,D\phi(x_0)) + m(\bar{\tau}) \end{cases} \tag{6.23}$$

To see this, observe that, if for some subsequence (which for simplicity is called again δ) $\delta \downarrow 0$, $\tau_0 \to \bar{\bar{\tau}}$, then, in view of (6.22), we have

$$m(\tau_0) + 2\delta - n(\tau_0) > \Psi(x_0,\tau_0) > \Psi(x,\tau) >$$
$$> (u(x,\tau) - \phi(x) + (\tau - t)H_1(\bar{t},x,D\phi(x)))^+ - n(\tau)$$

therefore

$$m(\tau_0) - n(\tau_0) + 2\delta > \Psi(x_0,\tau_0) > m(\tau) - n(\tau) \quad \text{for every} \quad \tau \in I .$$

This implies that

$$m(\bar{\bar{\tau}}) - n(\bar{\bar{\tau}}) > m(\tau) - n(\tau)$$

and thus $\bar{\bar{\tau}} = \bar{\tau}$. Moreover, for every $\tau \in I$

$$m(\tau_0) + 2\delta - n(\tau_0) >$$
$$> (u(x_0,\tau_0) - \phi(x_0) + (\tau_0 - t)H_1(\bar{t},x_0,D\phi(x_0)))^+ - n(\tau_0) + 2\delta > m(\tau) - n(\tau)$$

therefore for $\tau = \bar{\tau}$

$$m(\bar{\tau}) - n(\bar{\tau}) > \overline{\lim_{\delta\downarrow 0}}\, (u(x_0,\tau_0) - \phi(x_0) + (\tau_0 - t)H_1(\bar{t},x_0,D\phi(x_0)))^+ - n(\bar{\tau}) >$$
$$> \underline{\lim_{\delta\downarrow 0}}\, (u(x_0,\tau_0) - \phi(x_0) + (\tau_0 - t)H_1(\bar{t},x_0,D\phi(x_0)))^+ - n(\bar{\tau}) >$$
$$> m(\bar{\tau}) - n(\bar{\tau})$$

and thus (6.23), since $m(\bar{\tau}) > 0$.

Now since $(x_0,\tau_0) \in \mathbb{R}^N \times (t,\bar{t})$ is a local maximum of

$$(x,\tau) \to u(x,\tau) - \phi(x) + (\tau - t)H_1(\bar{t},x,D\phi(x)) + 2\delta\zeta(x) - n(\tau)$$

in view of (1.1), we have

$$n'(\tau_0) - H_1(\bar{t},x_0,D\phi(x_0)) + H_1(\tau_0,x_0,D\phi(x_0) -$$
$$- (\tau_0 - t)DH_1(\bar{t},\cdot,D\phi(\cdot))(x_0) - 2\delta D\zeta(x_0)) \leq 0$$

therefore

$$n'(\tau_0) \leq N_1(\bar{t} - \tau_0)(1 + \|D\phi\|) + M_1(\tau_0 - t)|DH_1(\bar{t},\cdot,D\phi(\cdot))(x_0)| + M_1 2\delta$$

where N_1, M_1 are defined by (H6) and (H7) respectively. Moreover, in view of (H4) and (H7), we have

$$|DH_1(\bar{t},\cdot,D\phi(\cdot))(x_0)| \leq \bar{C}_1(1 + \|D\phi\|) + M_1\|D^2\delta\|$$

where $\bar{C}_1$ is given by (H4). Combining all the above and letting $\delta \downarrow 0$, we obtain

$$n'(\bar{\tau}) \leq (\bar{t} - t)(N_1 + M_1\bar{C}_1 + (M_1)^2)(1 + \|D\phi\| + \|D^2\phi\|)$$

thus the result.

(b) Here we use

(6.24) $$m(\tau) = \|(u_2(\cdot,\tau) - \phi + (\tau - t)H_2(\bar{t},\cdot,D\phi) + (\bar{t} - t)H_1(\bar{t},\cdot,D\phi))^+\|$$

Since the proof is similar to the one given in (a), we omit it.

Now we continue with the proof of Theorem 6.1(b). In view of (6.14), (6.16) and the way that H_i^n are chosen, we have

$$\left\|\frac{F(t,\rho,u,\phi) - \phi}{\rho} + (H_1 + H_2)(t,\cdot,u,D\phi)\right\| \leq \Gamma_2(1 + \|D\phi\| + \|D^2\phi\|)\rho +$$
$$+ 2\sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\leq C}} |\tilde{H}_i^n(\tau,x,p) - H_i(\tau,x,u(x),p)|$$

where Γ_2 is a constant which depends only on $\|u\|$ and $H_i(i = 1,2)$. Letting $n \to \infty$ implies the result.

To prove (6.10) in the case that u_p is defined by (2.42) for

(6.25) $$F(t,\rho,v) = U_2(t,t - \rho)U_1(t,t - \rho)v$$

we need to check the assumptions of Theorem 2.2 (a). The fact that $F(t,\rho,v) \in C_b^{0,1}(\mathbb{R}^N)$

and (F15) are immediate consequences of Propositions 1.4 and 1.8. Moreover, for every $(t,\rho) \in K$ and $u,v \in C_b^{0,1}(\mathbb{R}^N)$, let

$$\bar{F}(t,\rho,u,v) = U_2(t,t-\rho,u)U_1(t,t-\rho,u)v$$

The only assumption we need to check is (F16). To this end, let $u \in C_b^{0,1}(\mathbb{R}^N)$. It is easy to check, using (1.1), (1.2), that $U_1(\tau,\tau-\rho)u(x)$ and $U_2(\tau,\tau-\rho)U_1(t,t-\rho)u(x)$ are viscosity solutions of the problems

$$\begin{cases} \dfrac{\partial u_2}{\partial \tau} + \tilde{H}_2(\tau,x,Du_2) = 0 & \text{in } \mathbb{R}^N \times (t-\rho,t] \\ u_2(x,t-\rho) = U_1(t,t-\rho)u(x) & \text{in } \mathbb{R}^N \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial u_1}{\partial \tau} + \tilde{H}_1(\tau,x,Du_1) = 0 & \text{in } \mathbb{R}^N \times (t-\rho,t] \\ u_1(x,t-\rho) = u \end{cases}$$

respectively, where

$$\tilde{H}_2(\tau,x,p) = H_2(\tau,x,U_2(\tau,\tau-\rho)U_1(t,t-\rho)u(x),p)$$

and

$$\tilde{H}_1(\tau,x,p) = H_1(\tau,x,U_1(\tau,\tau-\rho)u(x),p).$$

Moreover, in view of Propositions 1.5 and 1.8, there is a constant $A = A(\|u\|,\|Du\|)$ such that, for every $(t,\rho),(\tau,\sigma) \in K$, it is

$$\|DU_2(\tau,\tau-\sigma)U_1(t,t-\rho)u\|,\|DU_1(t,t-\rho)u\| \le A$$

and

$$\|DU_2(\tau,\tau-\sigma,u)U_1(t,t-\rho,u)u\|,\|DU_1(t,t-\rho,u)u\| \le A$$

It follows from Proposition 1.4 that

$$\|F(t,\rho,u) - \bar{F}(t,\rho,u,u)\| \le \|U_1(t,t-\rho)u - U_1(t,t-\rho,u)u\| +$$

$$+ \rho \sup_{\substack{x\in\mathbb{R}^N \\ |p|\le A \\ \tau\in[t-\rho,t]}} |H_2(\tau,x,U_2(\tau,\tau-\rho)U_1(t,t-\rho)u(x),p) - H_2(\tau,x,u(x),p)|$$

$$\le \rho(\bar{L}_2 \sup_{\tau\in[t-\rho,t]} \|U_2(\tau,\tau-\rho)U_1(t,t-\rho)u-u\| + \bar{L}_1 \sup_{t\in[t-\rho,t]} \|U_1(\tau,\tau-\rho)u-u\|)$$

Finally, (1.16) implies that

$$(6.26)\qquad \|F(t,\rho,u) - \bar{F}(t,\rho,u,u)\| \le \rho^2 e^{T(\bar{L}_1+\bar{L}_2)}\Big((\bar{L}_2)^2 \sup_{\substack{(x,\tau)\in\bar{Q}_T\\ |p|\le A\\ |r|\le R}} |H_2(\tau,x,r,p)| +$$

$$+ \bar{L}_1(\bar{L}_2 + \bar{L}_1) \sup_{\substack{(x,\tau)\in\bar{Q}_T\\ |p|\le A\\ |r|\le\|u\|}} |H_1(\tau,x,r,p)|\Big)$$

where $R = e^{T(\bar{L}_2+\bar{L}_1)}(\|u\| + T(C_2 + C_1))$.

(a) To verify (6.9) in the case that u_p is defined by (2.42) and

$$F(t,\rho,u) = U_2(t,t-\rho)U_1(t,t-\rho)u$$

we first assume that H_i satisfies (H5) with constant $\bar{L}_i$ independent of R. Then it suffices to check the hypotheses of Theorem 2.2 (b). In particular, by Proposition 1.4 we have

$$\|F(t,\rho,u) - F(t,\rho,\bar{u})\| \le e^{\rho(\bar{L}_1+\bar{L}_2)}\|u - \bar{u}\| .$$

Next, and in view of Remark 2.3, let $\bar{F}(t,\rho,\cdot,\cdot) : C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ be defined by

$$\bar{F}(t,\rho,u,v) = U_2(t,t-\rho,u)U_1(t,t-\rho,u)v$$

Using the arguments of the first part of (b) it is easy to see that $\bar{F}(t,\rho,\cdot,\cdot)$ satisfies (F1), (F2), (F3), (F4), (F10), (F11). For (F9) observe, that if $u,\bar{u},v,\bar{v} \in C_b^{0,1}(\mathbb{R}^N)$, then

$$\|F(t,\rho,u,v) - F(t,\rho,\bar{u},\bar{v})\| \le \|F(t,\rho,u,v) - F(t,\rho,u,\bar{v})\| +$$
$$+ \|F(t,\rho,u,\bar{v}) - F(t,\rho,\bar{u},\bar{v})\|$$

It follows from Proposition 1.4 that

$$\|F(t,\rho,u,v) - F(t,\rho,u,\bar{v})\| = \|U_2(t,t-\rho,u)U_1(t,t-\rho,u)v -$$
$$- U_2(t,t-\rho,u)U_1(t,t-\rho,u)\bar{v}\| \le \|U_1(t,t-\rho,u)v - U_1(t,t-\rho,u)\bar{v}\| \le \|v - \bar{v}\|$$

Moreover, if $R > \max(\|\bar{v}\| + \rho(L_1 + \bar{L}_2)\|u\| + \rho(C_1 + C_2), \|\bar{v}\| + \rho(\bar{L}_1 + \bar{L}_2)\|\bar{u}\| + \rho(C_1 + C_2))$, then by Proposition 1.5 (a) and Proposition 1.4, for $\varepsilon > 0$ and β_ε as in (1.8), we have

$$\sup_{|x-y|\le\varepsilon} \{|F(t,\rho,u,\bar{v})(x) - F(t,\rho,\bar{u},\bar{v})(x)| + 3R\beta_\varepsilon(x-y)\} \le$$

$$\le \sup_{|x-y|\le\varepsilon} \{|U_1(t,t-\rho,u)\bar{v}(x) - U_1(t,t-\rho,\bar{u})v(y)| + 3R\beta_\varepsilon(x-y)\}$$

$$+\rho \sup_{\substack{|x-y|\le\varepsilon\\ |p|\le B\\ |r|\le R\\ \tau\in[0,T]}} |H_2(\tau,x,u(x),p) - H_2(\tau,y,\bar{u}(y),p)| \le$$

$$\le \sup_{|x-y|\le\varepsilon} \{|\bar{v}(x) - \bar{v}(y)| + 3R\beta_\varepsilon(x-y)\} +$$

$$+\rho\Big[\sup_{\substack{|x-y|\le\varepsilon\\ |p|\le B\\ |r|\le R\\ \tau\in[0,T]}} |H_2(\tau,x,u(x),p) - H_2(\tau,y,\bar{u}(y),p)| +$$

$$+\sup_{\substack{|x-y|\le\varepsilon\\ |p|\le B\\ |r|\le R\\ \tau\in[0,T]}} |H_1(\tau,x,u(x),p) - H_1(\tau,y,\bar{u}(y),p)|\Big] .$$

where B is such that

$$\sup_{\tau\in[t-\rho,t]} \|DU_1(\tau,\tau-\rho,\bar{u})\bar{v}\|, \quad \sup_{\tau\in[t-\rho,t]} \|DU_2(\tau,\tau-\rho,\bar{u})U_1(t,t-\rho,\bar{u})\bar{v}\| \le B .$$

Then, in view of (H4), (H5), the above inequality implies

$$\|F(t,\rho,u,\bar{v}) - F(t,\rho,\bar{u},v)\| \le \|D\bar{v}\|\varepsilon + \rho(\bar{L}_1 + \bar{L}_2)\|D\bar{u}\|\varepsilon$$

$$+\rho(\bar{L}_1 + \bar{L}_2)\|D\bar{u}\|\varepsilon + \rho\big(C^1_{R_1} + C^2_{R_1}\big)(1+|B|)\varepsilon$$

where for $R_1 = \max(\|u\|,\|\bar{u}\|)$ and $i = 1,2$, $C^i_{R_1}$ is giver by (H4). Letting $\varepsilon \to 0$ above we obtain

$$\|F(t,\rho,u,\bar{v}) - F(t,\rho,\bar{u},\bar{v})\| \le \rho(\bar{L}_1 + \bar{L}_2)\|u - \bar{u}\|$$

and thus the result. Moreover, we want to verify (F12). To this end, for $u \in C^{0,1}_b(\mathbb{R}^N)$, $\phi \in C^2_b(\mathbb{R}^N)$ and with the notation used for the verification of (F8) in (b), we have

$$\Big\|\frac{F(t,\rho,u,\phi)-\phi}{\rho} + (H_1 + H_2)(t,\cdot,u,D\phi)\Big\| \le 2\sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T\\ |p|\le\bar{C}}} |\tilde{H}^n_i(\tau,x,p) - H_i(\tau,x,u(x),p)|$$

$$+\Big\|\frac{\tilde{U}^n_2(t,t-\rho)\tilde{U}^n_1(t,t-\rho)\phi-\phi}{\rho} + (\tilde{H}_1 + \tilde{H}_2)(t,\cdot,D\phi)\Big\|$$

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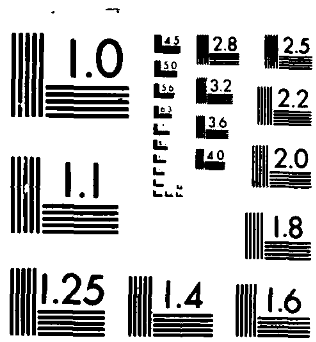
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where now for $i = 1,2$, $\tilde{H}_i^n$ satisfy (H1), (H2), (H4), (H5), (H6) and (H7) with constants depending on n. For $\eta > 0$ choose n large enough, so that

$$2 \sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le C}} |\tilde{H}_i^n(\tau,x,p) - H_i(\tau,x,u(x),p)| < \frac{\eta}{2}$$

In view of Lemma 4.2 we have

$$\left\| \frac{\tilde{U}_2^n(t,t-\rho)\tilde{U}_1^n(t,t-\rho)\phi - \phi}{\rho} + (\tilde{H}_1^n + \tilde{H}_2^n)(t,\cdot,D\phi) \right\| \le \Gamma_2^n(1 + \|D\phi\| + \|D^2\phi\|)\rho$$

If ρ is such that

$$\Gamma_2^n(1 + \|D\phi\| + \|D^2\phi\|)\rho < \frac{\eta}{2}$$

then

$$\left\| \frac{F(t,\rho,u,\phi) - \phi}{\rho} + (H_1 + H_2)(t,\cdot,u,D\phi) \right\| < \eta$$

which implies (F12). Finally (F16), and therefore (F17), is proved here the same way as in (b).

To prove (6.9) in the case that H_1,H_2 satisfy (H3), with constants $\gamma_1,\gamma_2 \le 0$ independent of R, observe that it suffices to assume $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. Indeed, for every $u_0 \in BUC(\mathbb{R}^N)$ we can find a sequence $\{u_0^n\}$ in $C_b^{0,1}(\mathbb{R}^N)$, such that $\|u_0^n - u_0\| \to 0$ as $n \to \infty$. But, in view of Proposition 1.4, it is easy to see that, if $u_P^n : \bar{Q}_T \to \mathbb{R}$ is defined by (2.42) for initial value u_0^n and u^n is the viscosity solution of (0.1) in $\bar{Q}_T$ for $H = H_1 + H_2$, then

$$\|u - u^n\| \le e^{-T(\gamma_1+\gamma_2)}\|u_0 - u_0^n\|$$

and

$$\|u_P - u_P^n\| \le e^{-T(\gamma_1+\gamma_2)}\|u_0 - u_0^n\| .$$

Then

$$\|u_P - u\| \le 2e^{-T(\gamma_1+\gamma_2)}\|u_0 - u_0^n\| + \|u^n - u_P^n\| \tag{6.26}$$

and thus the claim is proved.

Next we assume that $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. By Proposition 1.5 (a) if $R > 0$ is such that

$$e^{-(\gamma_1+\gamma_2)T}(\|u_0\| + T(C_1 + C_2 + 2)) < R$$

then

$$\|u\|, \|u_p\| < R$$

where, for $i = 1,2$, C_i is given by (H2). Moreover, if, for $i = 1,2$, $\bar{C}_i = C_R^i$ is given by (H4), then, by Propositions 1.5 (c) and 1.8,

$$\sup_{0 \le \tau \le T} \|Du(\cdot,\tau)\|, \ \sup_{0 \le \tau \le T} \|Du_p(\cdot,\tau)\| < C$$

where $\bar{C} = e^{(4(\bar{C}_1+\bar{C}_2)e^{-(\gamma_1+\gamma_2)T} - (\gamma_1+\gamma_2))T}(\|Du_0\| + 2T(\bar{C}_1 + \bar{C}_2))$. Now for $i = 1,2$ consider $H_i^n : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying (H1), (H2), (H4), (H5) and such that

(i) $H_i^n \to H_i$ as $n \to \infty$ uniformly on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,\bar{C})$

(ii) The constant C_i^n in (H2) is such that $C_i^n < C_i + 1$.

(iii) The constant $C_R^{i,n}$ in (H5) is such that $C_R^{i,n} < 2C_R^i$ for $R > 0$

(iv) H_i^n satisfies (H3) with the same constant as H_i.

If u^n is the viscosity solution of (0.1) in $\bar{Q}_T$ for $H^n = H_1^n + H_2^n$ and if $u_p^n : \bar{Q}_T \to \mathbb{R}$ is defined by (2.42) using

$$F^n(t,\rho,u) = U_2^n(t,t-\rho)U_1^n(t,t-\rho)u$$

where U_i^n correspond to H_i^n, then by Proposition 1.5 it is

$$\|u^n\|, \|u_p^n\| < R$$

and

$$\sup_{0 \le \tau \le T} \|Du^n(\cdot,\tau)\|, \ \sup_{0 \le \tau \le T} \|Du_p^n(\cdot,\tau)\| < \bar{C}$$

Moreover, in view of Proposition 1.4, for $\varepsilon > 0$ and β_ε as in (1.8) we have

$$\sup_{|x-y|<\varepsilon} \{|u(x,\tau) - u^n(y,\tau)| + 3Re^{-(\gamma_1+\gamma_2)\tau}\beta_\varepsilon(x-y)\} <$$

$$< e^{-(\gamma_1+\gamma_2)\tau} \sup_{|x-y|<\varepsilon} \{|u_0(x) - u_0(y)| + 3R\beta_\varepsilon(x-y)\} +$$

$$+ e^{-(\gamma_1+\gamma_2)\tau}\tau \sup_{\substack{|x-y|<\varepsilon \\ t\in[0,T] \\ |r|<R \\ |p|<\bar{C}}} |(H_1 + H_2)(t,x,r,p) - (H_1^n + H_2^n)(t,y,r,p)|$$

and

$$\sup_{|x-y|<\varepsilon} \{|u_p(x,\tau) - u_p^n(x,\tau)| + 3Re^{-(\gamma_1+\gamma_2)\tau}\beta_\varepsilon(x-y)\} <$$

$$< e^{-(\gamma_1+\gamma_2)\tau} \sup_{|x-y|<\varepsilon} \{|u_0(x) - u_0(y)| + 3R\beta_\varepsilon(x-y)\} +$$

$$+ \tau e^{-(\gamma_1+\gamma_2)\tau} \sum_{i=1}^{n} \sup_{\substack{|x-y|<\varepsilon \\ t\in[0,T] \\ |r|<R \\ |p|<\bar{C}}} |H_i(t,x,r,p) - H_i^n(t,x,r,p)|$$

where the last inequality is proved by a simple inductive argument. The above then imply

$$\|u - u^n\| < Te^{-(\gamma_1+\gamma_2)T} \sum_{i=1}^{n} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|<R \\ |p|<\bar{C}}} |H_i(t,x,r,p) - H_i^n(t,x,r,p)| +$$

$$+ e^{-(\gamma_1+\gamma_2)T}(\|Du_0\| + 2(\bar{C}_1 + \bar{C}_2)(1 + \bar{C})T)\,\varepsilon$$

and

$$\|u_p - u_p^n\| < Te^{-(\gamma_1+\gamma_2)T} \sum_{i=1}^{n} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|<R \\ |p|<\bar{C}}} |H_i(t,x,r,p) - H_i^n(t,x,r,p)| +$$

$$+ e^{-(\gamma_1+\gamma_2)T}(\|Du_0\| + 2(\bar{C}_1 + \bar{C}_2)(1 + \bar{C})T)\varepsilon$$

Letting $\varepsilon \to 0$ we obtain

$$\|u_\rho - u\| \le \|u_\rho^n - u^n\| + 2Te^{-(\gamma_1+\gamma_2)T} \sum_{i=1}^{n} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le R \\ |p|\le \bar{C}}} |H_i(t,x,r,p) - H_i^n(t,x,r,p)|$$

For $\eta > 0$ let n be such that

$$2Te^{-(\gamma_1+\gamma_2)T} \sum_{i=1}^{n} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le R \\ |p|\le \bar{C}}} |H_i(t,x,r,p) - H_i^n(t,x,r,p)| < \frac{\eta}{2}$$

Since H_i^n satisfy (H5) we know that $\|u_\rho^n - u^n\| \to 0$ as $|P| \to 0$. If ρ_0 is such that, if $|P| < \rho_0$, then

$$\|u_\rho^n - u_\rho\| < \eta/2$$

then, by the choice of n,

$$\|u_n - u\| < \eta$$

and thus the result.

<u>Remark 6.2</u>. Using the ideas involved in the proof of (a) above one can prove that, if for $i = 1,2$, H_i^n satisfy (H1), (H2), (H4) and (H5) and u_P is defined by (2.1 using $F(t,\rho,w,v) = U_2(t,t-\rho,w)U_1(t,t-\rho,w)v$, then $\|u_\rho - u\| \to 0$ as $|P| \to 0$. Since the proof is almost the same we omit it.

<u>Remark 6.3</u>. One can formulate a convergence theorem for schemes, for which we cannot verify directly the conditions of Theorems 2.1 and 2.2 (for example Theorem 6.1 (a)). In particular, we have to assume that for given H and F, we can find a sequence $\{H_\ell, F_\ell\}$, which satisfies the assumptions of Theorems 2.1 and 2.2 and, moreover, converges in a suitable sense to H and F. This is exactly what was done in the proof of Theorem 6.1 (a).

The second theorem of this section is concerned with the convergence of "resolvent"-type Trotter products, i.e. products formed by (6.6) or (6.8). In particular, we have

Theorem 6.2. (a) For $i = 1,2$, let $H_i : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1), (H2), (H4) (with constant C_i independent of R) and (H3) (with constant $\gamma_i < 0$ independent of R). For $u_0 \in BUC(\mathbb{R}^N)$, let $u \in BUC(\bar{Q}_T)$ be the viscosity solution of (0.1) in $\bar{Q}_T$ with $H = H_1 + H_2$. If, for a partition P of $[0,T]$, $u_P : \bar{Q}_T \to \mathbb{R}$ is defined by (2.42) using $F(t,\rho,v) = J_2(t,\rho)J_1(t,\rho)v$, then

(6.27) $$\|u_P - u\| \to 0 \quad \text{as} \quad |P| \to 0$$

(b) Suppose that, for $i = 1,2$, H_i satisfies (H1), (H2), (H4), (H5), (H6) and (H7) with constants independent of R. If $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ and for a partition P of $[0,T]$ $u_P : \bar{Q}_T \to \mathbb{R}$ is defined by either (2.1) using $F(t,\rho,w,v) = J_2(t,\rho,w)J_1(t,\rho,w)v$ or (2.42) using $F(t,\rho,v) = J_2(t,\rho)J_1(t,\rho)v$, then there exists a constant K depending only on $\|u_0\|$ and $\|Du_0\|$, such that

(6.28) $$\|u_P - u\| \le K|P|^{1/2}$$

for $|P|$ sufficiently small.

Remark 6.4. A remark analogous to Remark 6.1 applies to Theorem 6.2 too.

Proof of Theorem 6.2. (b) We first prove (6.28) in the case that u_P is defined by (2.1) for

$$F(t,\rho,w,v) = J_2(t,\rho,w)J_1(t,\rho,w)v$$

To this end, it suffices to check the assumptions of Theorem 2.1 (a). In view of Proposition 1.9, if $w,v \in C_b^{0,1}(\mathbb{R}^N)$, then $F(t,\rho,w,v) \in C_b^{0,1}(\mathbb{R}^N)$, provided that ρ is sufficiently small. Moreover, since for $i = 1,2$ $J_i(t,0,w)v = v$ for every $w,v \in C_b^{0,1}(\mathbb{R}^N)$, it is immediate that (F1) is satisfied. (F2) follows from Propositions 1.6 and 1.8. In particular, if $A > 0$ such that

$$\|DF(t,\rho,u,u)\|, \|DJ_1(t,\rho,u)u\| \le A$$

for $(t,\rho) \in K$ (such an A exists by Proposition 1.8), then

$$\|F(t,\rho,u,u) - F(\bar t,\bar\rho,u,u)\| \le \|J_2(t,\rho,u)J_1(t,\rho,u)u - J_2(\bar t,\bar\rho,u)J_1(t,\rho,u)u\| +$$

$$+ \|J_2(\bar t,\bar\rho,u)J_1(t,\rho,u)u - J_2(\bar t,\bar\rho,u)J_1(\bar t,\bar\rho,u)u\|$$

$$+ \sum_{i=1}^{2} \rho \sup_{\substack{(x,t)\in\bar Q_T \\ |p|\le A}} |H_i(t,x,u(x),p) - H_i(\bar t,x,u(x),p)| +$$

$$+ |\rho - \bar\rho| \sup_{\substack{(x,\tau)\in\bar Q_T \\ |p|\le A}} |H_i(\tau,x,u(x),p)|$$

and thus the claim. (F3) is an immediate consequence of the definition of the viscosity solution. Next, and in view of (1.17), for $u,v,\bar v \in C_b^{0,1}(\mathbb{R}^N)$ it is

$$\|F(t,\rho,u,v) - F(t,\rho,u,\bar v)\| \le \|v - \bar v\|$$

which, by the discussion after the statement of Theorem 2.1, implies (F5) for $\bar L = \infty$. Moreover, in view of Proposition 1.7, $u \in C_b^{0,1}(\mathbb{R}^N)$ we have

$$\|F(t,\rho,u,u)\| \le \|J_1(t,\rho,u)u\| + \rho(C_2 + \bar L_2\|u\|) \le \|u\| + \rho(\bar L_1 + \bar L_2)\|u\| + \rho(C_1 + C_2)$$

and thus (F6), where for $i = 1,2$, $C_i,\bar L_i$ are given by (H2) and (H5) respectively. If, for $i = 1,2$, $\bar C_i$ is given by (H4), then, by Proposition 1.7 and for $\rho(\bar C_1 + \bar C_2) \le \frac{1}{2}$, it is

$$\|DJ_1(t,\rho,u,u)\| \le e^{(2\bar C_1+\bar L_1)\rho}(\|Du\| + \rho\bar C_1)$$

and

$$\|DF(t,\rho,u,u)\| \le e^{(2\bar C_2+\bar L_2)\rho}(e^{(2\bar C_1+\bar L_1)\rho}(\|Du\| + \rho\bar C_1) + \rho\bar C_2)$$

i.e.

$$\|DF(t,\rho,u,u)\| \le e^{[2(\bar C_2+\bar C_1)+(\bar L_1+\bar L_2)]\rho}(\|Du\| + \rho(\bar C_1 + \bar C_2))$$

which proves (F7). Proposition 1.7 also implies F4, since

$$\|F(t,\rho,u,u) - u\| \le \rho \sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\mathbb{R}^N \\ |p|\le A}} |H_i(\tau,x,u(x),p)|$$

where A is as in (6.30). Finally we need to check (F8). To this end, for $i = 1,2$,

consider smooth functions $\tilde{H}_i^n : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ which satisfy the same conditions as H_i, with constants depending on the constants of H_i and $\|u\|, \|Du\|$ and, moreover,

$$\begin{cases} \tilde{H}_i^n(t,x,p) \to H_i(t,x,u(x),p) \quad \text{as} \quad n \to \infty \\ \text{uniformly on} \quad [0,T] \times \mathbb{R}^N \times B_N(0,R) \quad \text{for each} \quad R > 0. \end{cases}$$

Then, by the previous discussion and Proposition 1.5, $\tilde{J}_2^n(t,\rho)\tilde{J}_1^n(t,\rho)\phi \in C_b^{0,1}(\mathbb{R}^N)$, where $\tilde{J}_i^n$ corresponds to $\tilde{H}_i^n$ by (6.6) and $\phi \in C_b^2(\mathbb{R}^N)$. Moreover, there exists a constant $\bar{\bar{C}}$ depending only on $\|u\|, \|Du\|, H_i, \|\phi\|$ and $\|D\phi\|$ such that

(6.31) $\quad \|DJ_2(\tau,\sigma,u)J_1(t,\rho,u)\phi\|,\ \|DJ_1(t,\rho,u)\phi\|,\ \|D\tilde{J}_2^n(\tau,\sigma)\tilde{J}_1^n(t,\rho)\phi\|,\ \|D\tilde{J}_1^n(t,\rho)\phi\| \le \bar{\bar{C}}$

for every $(\tau,\sigma),(t,\rho) \in K$. Then, in view of Proposition 1.6, we have

$$\|F(t,\rho,u,\phi) - \tilde{J}_2^n(t,\rho)\tilde{J}_1^n(t,\rho)\phi\| \le \rho \sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le\bar{\bar{C}}}} |\tilde{H}_i^n(\tau,x,p) - H_i(\tau,x,u(x),p)| \tag{6.32}$$

and therefore

$$\begin{aligned} &\left\|\frac{F(t,\rho,u,\phi) - \phi}{\rho} + (H_1 + H_2)(t,\cdot,u,D\phi)\right\| \le \\ &\le 2 \sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le\bar{\bar{C}}}} |\tilde{H}_i^n(\tau,x,p) - H_i(\tau,x,u(x),p)| \\ &+ \left\|\frac{\tilde{J}_2^n(t,\rho)\tilde{J}_1^n(t,\rho)\phi - \phi}{\phi} + (\tilde{H}_1 + \tilde{H}_2)(t,\cdot,D\phi)\right\| \end{aligned} \tag{6.33}$$

In order to finish we need the following lemma

<u>Lemma 6.2</u>. (a) Let $H_1 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be smooth and assume that it satisfies (H1), (H2), (H4), (H5), (H6) and (H7) with constants independent of R. If, for $\phi \in C_b^2(\mathbb{R}^N)$, $u_1 \in BUC(\mathbb{R}^N)$ is the viscosity solution of

$$u_1 + \rho H_1(t,x,Du_1) = \phi \quad \text{in} \quad \mathbb{R}^N$$

then

$$\|u_1 - \phi + \rho H_1(t,\cdot,D\phi)\| \le \rho^2 \Gamma_3(1 + \|D\phi\| + \|D^2\phi\|) \tag{6.34}$$

where Γ_3 depends only on the constants related with H_1 by (H4), (H5), (H7).

(b) Let $H_2 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfy the same hypotheses as H_1 in (a). If $u_2 \in BUC(\mathbb{R}^N)$ is the viscosity solution of

$$u_2 + \rho H_2(t,x,Du_2) = u_1 \quad \text{in} \quad \mathbb{R}^N$$

then

(6.35) $$\|u_2 - \phi + \rho(H_1(t,\cdot,D\phi) + H_2(t,\cdot,D\phi)\| \leq \rho^2\Gamma_4(1 + \|D\phi\| + \|D^2\phi\|)$$

where Γ_4 is a constant, which depends only on the constants related with H_1,H_2 by (H4), (H5), (H7).

<u>Proof</u>. (a) Here, as usual, we only show

$$\|(u_1 - \phi + \rho H_1(t,\cdot,D\phi))^+\| \leq \rho^2\Gamma_1(1 + \|D\phi\| + \|D^2\phi\|)$$

To this end, without loss of generality, we assume

$$\|(u_1 - \phi + \rho H_1(t,\cdot,D\phi))^+\| > 0$$

and we define $\Phi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Phi(x) = (u_1(x) - \phi(x) + \rho H_1(t,x,D\phi(x)))^+$$

Since Φ is bounded, for every $\delta > 0$ there is a point $x_1 \in \mathbb{R}^N$, such that

$$\Phi(x_1) > \sup_{x\in\mathbb{R}^N} \Phi(x) - \delta$$

Next choose $\zeta \in C_0^\infty(\mathbb{R}^N)$ so that $0 \leq \zeta \leq 1$, $|D\zeta| \leq 1$, $|D^2\zeta| \leq 2$, $\zeta(x_1) = 1$ and define $\Psi : \mathbb{R}^N \to \mathbb{R}$ by

(6.37) $$\Psi(x) = \Phi(x) + 2\delta\zeta(x)$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1) > \sup_{x\in\mathbb{R}^N} \Phi(x) + \delta$$

there is a point $x_0 \in \mathbb{R}^N$ such that

$$\Psi(x_0) \geq \Psi(x) \quad \text{for every} \quad x \in \mathbb{R}^N$$

We claim that x_0 satisfies

(6.39) $$\begin{cases} \text{As} \quad \delta \to 0 \\ (u_1(x_0) - \phi(x_0) + \rho H_1(t,x_0,D\phi(x_0)))^+ = u(x_0) - \phi(x_0) + \rho H_1(t,x_0,D\phi(x_0)) \\ \longrightarrow \sup_{x\in\mathbb{R}^N}(u_1(x) - \phi(x) + \rho H_1(t,x,D\phi(x)))^+ \end{cases}$$

To see this observe that, in view of (6.38), we have

$$\|(u_1 - \phi + \rho H_1(t,\cdot,D\phi))^+\| \geq \overline{\lim_{\delta\to 0}}\,(u_1(x_0) - \phi(x_0) + \rho H_1(t,x_0,D\phi(x_0)))^+ \geq$$

$$\geq \underline{\lim_{\delta\to 0}}\,(u_1(x_0) - \phi(x_0) + \rho H_1(t,x_0,D\phi(x_0)))^+ \geq \|(u_1 - \phi + \rho H_1(t,\cdot,D\phi))^+\|$$

and thus (6.39).

Now since $x_0 \in \mathbb{R}^N$ is a local maximum of $x \to u_1(x) - \phi(x) + \rho H_1(t,x,D\phi(x)) + 2\delta\zeta(x)$ in view of (1.3), it is

$$u_1(x_0) + \rho H_1(t,x_0,D\phi(x_0)) - \rho DH_1(t,\cdot,D\phi)(x_0) - 2\delta D\zeta(x_0)) \leq \phi(x_0)$$

therefore

$$\begin{aligned} u_1(x_0) - \phi(x_0) + \rho H_1(t,x_0,D\phi(x_0)) &\leq \\ &\leq \rho(H_1(t,x_0,D\phi(x_0)) - H_1(t,x_0,D\phi(x_0) - \rho DH_1(t,\cdot,D\phi)(x_0) - 2\delta D\zeta(x))) \\ &\leq \rho M_1(\rho(C_1 + M_1)(1 + \|D\phi\| + \|D^2\phi\|) + 2\delta) \end{aligned}$$

where $\bar{C}_1, M_1$ are given by (H4), (H7) respectively. Letting $\delta \to 0$ above implies (6.36) with

$$\Gamma_3 = M_1(\bar{C}_1 + M_1) .$$

(b) Here we define

$$\Phi(x) = (u_2(x) - \phi(x) + \rho(H_1(t,x,D\phi(x)) + H_2(t,x,D\phi(x))))^+$$

Since the proof is similar to the one of (a), we omit it.

Now we continue with the proof of Theorem 6.2 (b). In view of (6.33), (6.35) and the way that $\tilde{H}_i^n$ are chosen, we have

$$\left\|\frac{F(t,\rho,u,\phi) - \phi}{\rho} + (H_1 + H_2)(t,\cdot,u,D\phi)\right\| \leq \Gamma_4(1 + \|D\phi\| + \|D^2\phi\|)\rho +$$

$$+ 2\sum_{i=1}^{2} \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\leq\bar{C}}} |H_i^n(\tau,x,p) - H_i(\tau,x,u(x),p)|$$

Letting $n \to \infty$ we obtain the result.

To prove (6.28) in the case that u_p is defined by (2.42) for

(6.40) $$F(t,\rho,v) = J_2(t,\rho)J_1(t,\rho)v$$

we need to check the assumptions of Theorem 2.2 (a). The fact that $F(t,\rho,v) \in C_b^{0,1}(\mathbb{R}^N)$

and (F15) are immediate consequences of Propositions 1.7 and 1.8. Moreover, for every $(t,\rho) \in K$ and $u,v \in C_b^{0,1}(\mathbb{R}^N)$ let

$$\bar{F}(t,\rho,u,v) = J_2(t,\rho,u)J_1(t,\rho,u)v$$

In view of the previous discussion, the only assumption we need to check is (F16). To this end, let $u \in C_b^{0,1}(\mathbb{R}^N)$. It is easy to see, using the definitions, that $J_1(t,\rho)u$ and $J_2(t,\rho)J_1(t,\rho)u$ are viscosity solutions of the problems

$$u_2 + \tilde{H}_2(t,x,Du_2) = J_1(t,\rho)u \quad \text{and} \quad u_1 + \tilde{H}_1(t,x,Du_1) = u$$

respectively, where

$$\tilde{H}_1(t,x,p) = H_1(t,x,J_1(t,\rho)u(x),p)$$

and

$$\tilde{H}_2(t,x,p) = H_2(t,x,J_2(t,\rho)J_1(t,\rho)u(x),p) \ .$$

Moreover, by Propositions 1.6 and 1.8 (b), there is a constant $A = A(\|u\|,\|Du\|)$ such that

$$\|DF(t,\rho,u)\|,\|DJ_1(t,\rho,u)\|,\|D\bar{F}(t,\rho,u,u)\|,\|DJ_1(t,\rho,u,u)\| \le A$$

It follows from Proposition 1.6 that

$$\begin{aligned} \|F(t,\rho,u) - \bar{F}(t,\rho,u,u)\| &\le \|J_1(t,\rho)u - J_1(t,\rho,u)u\| + \\ &+ \rho \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le A}} |\tilde{H}_2(\tau,x,p) - H_2(\tau,x,u(x),p)| \\ &\le \rho(\bar{L}_2\|J_2(t,\rho)J_1(t,\rho)u - u\| + \bar{L}_1\|J_1(t,\rho)u - u\|) \end{aligned}$$

Finally Proposition 1.7 implies that

$$\begin{aligned} \|F(t,\rho,u) - \bar{F}(t,\rho,u,u)\| &\le \rho^2 e^{2T(\bar{L}_1+\bar{L}_2)}((\bar{L}_2)^2 \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le A \\ |r|\le R}} |H_2(\tau,x,r,p)| + \\ &+ \bar{L}_1(\bar{L}_2 + \bar{L}_1) \sup_{\substack{(x,\tau)\in\bar{Q}_T \\ |p|\le A \\ |r|\le R}} |H_1(\tau,x,r,p)|) \end{aligned}$$

where $R = e^{2T(\bar{L}_2+\bar{L}_1)}(\|u\| + T(C_1 + C_2))$

(a) The proof of (a) is almost identical with the proof (a) of Theorem 6.1 (a) with the appropriate modifications, therefore we omit it here.

Remark 6.5. A remark analogous to Remark 6.2 applies to Theorem 6.2 too.

REFERENCES

[1] Barles, G. To appear.

[2] Barron, N. E., L. C. Evans and R. Jensen: Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls. To appear in Journal of Differential Equations.

[3] Capuzzo Dolcetta, I. and L. C. Evans: Optimal switching for ordinary differential equations. To appear in SIAM Journal of Control and Optimization.

[4] Crandall, M. G., L. C. Evans and P. L. Lions: Some properties of viscosity solutions of Hamilton-Jacobi equations. Mathematics Research Center TSR #2390, University of Wisconsin-Madison, June 1982 and to appear in Trans. Amer. Math. Soc.

[5] Crandall, M. G. and P. L. Lions: Viscosity solutions of Hamilton-Jacobi equations. Mathematics Research Center TSR #2259, University of Wisconsin-Madison, August 1981 and to appear in Trans. Amer. Math. Soc.

[6] Crandall, M. G. and P. L. Lions: Two approximations of solutions of Hamilton-Jacobi equations. Mathematics Research Center TSR #2431, University of Wisconsin-Madison, September 1982.

[7] Crandall, M. G. and L. Tartar: Some relations between nonexpansive and order preserving mappings. Proc. Amer. Math. Soc. 79 (1979), 74-80.

[8] Elliott, R. J. and N. J. Kalton: The existence of value in differential games. Mem. AMS #126 (1972).

[9] Evans, L. C.: Some max-min methods for the Hamilton-Jacobi equation. To appear in Indiana U. Math. Journal.

[10] Evans, L. C. and P. E. Souganidis: Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. Mathematics Research Center TSR #2492, University of Wisconsin-Madison, March 1983.

[11] Fleming, W. H.: The convergence problem for differential games. J. Math. Analysis and Applications, 3 (1961) pp. 102-116.

[12] Fleming, W. H.: The convergence problem for differential games, II. In "Advances in Game Theory", Annals of Math. Study 52, pp. 195-210, Princeton University Press, Princeton 1964.

[13] Fleming, W. H.: The Cauchy problem for degenerate prarabolic equations. J. Math and Mechanics, 13 (1964) pp. 987-1008.

[14] Fleming, W. H.: Nonlinear partial differential equations probabilistic and game theoretic methods. In Problems in Nonlinear Analysis, CIME, Ed. Cremonese, Roma 1971.

[15] Friedman, A.: Differential Games, Wiley, New York, 1971.

[16] Friedman, A.: Differential Games, CBMS #18, AMS, Providence, 1974.

[17] Isaacs, R.: Differential Games, Wiley, New York, 1965.

[18] Lions, P. L.: Generalized Solutions of Hamilton-Jacobi Equations. Pitman, Boston, 1982.

[19] Lions, P. L.: Existence results for first-order Hamilton-Jacobi equations. To appear in Ricerche Math. Napoli, 1982-1983.

[20] Souganidis, P. E.: Existence of viscosity solutions of Hamilton-Jacobi equations. Mathematics Research Center TSR #2488, University of Wisconsin-Madison, March 1983.

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Equations of Hamilton-Jacobi type arise in many areas of application, including the calculus of variations, control theory and differential games. Recently M. G. Crandall and P. L. Lions established the correct notion of generalized solutions for these equations. This article discusses the convergence of general approximation schemes to this solution and gives,

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ABSTRACT (cont.)

under certain hypotheses, explicit error estimates. These results are then applied to obtain various representations. These include "max-min" representations of solutions relevant to the theory of differential games (which imply the existence of the "value" of the game), representations as limits of solutions of general explicit and implicit finite difference schemes, and as limits of several types of Trotter products.

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